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Risk-Sharing and Contagion Networks

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Abstract

We investigate the properties of financial networks that allow to optimally solve the trade-off between higher risk-sharing and contagion. With continuous shock distributions, the optimum features the segmentation of the system of firms into disjoint components, with uniform exposure within them. With positive mass on some large shocks, it is instead optimal to modulate the exposure level to different firms. When firms are heterogeneous in the risk characteristics of their shocks, optimality requires homogeneous components, while with heterogeneity in size, an irrelevance result holds. Also, the incentives of firms to establish linkages may not be aligned with social optimality.

Keywords: Firm networks, Contagion, Risk Sharing

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1 Introduction

Recent economic events have made it clear that looking at financial entities in isolation, abstracting from their linkages, gives an incomplete, and possibly very misleading, impression of the potential impact of shocks to the financial system. In the words of Acharya *et al.* (2010) “current financial regulations, such as Basel I and Basel II, are designed to limit each institution’s risk seen in isolation; they are not sufficiently focused on systemic risk even though systemic risk is often the rationale provided for such regulation.” The aim of this paper is precisely to investigate how the capacity of the system to absorb shocks depends on the pattern of interconnections established among firms.

More specifically, we study the tradeoff between the risk-sharing benefits to firms of higher interconnection and the costs resulting from an increased risk exposure. Clearly, this trade-off must be at the center of any regulatory effort of the financial world that takes a truly systemic view of the problem. By formulating the problem in a stylized and analytically tractable framework, we can examine how the segmentation of the system into separate components as well as the density of the connections within each component should be tailored to the underlying shock structure. It also sheds light on the key issue of whether the normative prescriptions on the optimal pattern of linkages are consistent with the individual incentives to form or remove links.

We analyze a finite environment with N financial firms. For simplicity, in most of the paper we shall consider the case where all firms are identical *ex ante*, with the same level of assets and liabilities, and endowed with a risky project displaying the same probabilistic pattern of returns. *Ex post*, however, they will be often different since we assume that, with some probability, a shock hits a randomly selected project, decreasing the income it generates. We allow each firm to hold claims to the yield of the projects of other firms, in addition to claims on its own project. This generates financial linkages among the firms in the system. It implies that a shock hitting the project of a firm also affects all other firms linked with it, in proportion to their exposure to this firm.

When the revenue of the assets of a firm falls below its liabilities a firm defaults. The presence of linkages to other firms allows a firm to reduce the exposure to the shocks that may hit its own project, and hence to lower the probability of default in such event, thus providing risk sharing. At the same time, these linkages expose the firm to shocks hitting other firms. The precise effects of the shock depend then on the pattern of financial linkages among all the firms, that is on the system’s financial structure.

Clearly, the maximum extent of risk sharing obtains when all firms belong to a fully connected network.

This configuration, however, exhibits the widest exposure of firms to shocks. Any shock hitting a firm's project affects all firms in the system and, if the shock is large, could lead to extensive default. There are two alternative (possibly complementary) ways of reducing the possibility of contagion. One is 'segmentation,' that is to have firms connected only to a subset of other firms; the system of firms is so divided into disjoint components¹ which are then isolated from the shocks hitting any other component. A second way is to operate on the 'density of the connections', that is to modulate the exposure that any given firm has to the other firms in its component. For example, a firm could have a high level of exposure to some firms, while keeping a significantly lower exposure to the remaining ones.

One of the primary objectives of this paper is to identify, for given characteristics of the shock distribution, the financial structure that maximizes social surplus, thus dealing optimally with the trade-off between risk sharing and contagion. In the environment considered social surplus is maximized when the expected number of defaults in the system is minimal.

More specifically, we find that the first way to limit contagion – i.e. using segmentation – proves to be optimal when firms face shocks whose distribution belongs to a canonical family that includes the set of Pareto (power-law) distributions, or mixtures of these distributions. In all these cases we find that it is always optimal to have uniform exposure among firms within components while the optimal segmentation varies with the likelihood of large shocks, that exceed the capacity of the system to absorb them. In particular, when the distribution of the shocks is Pareto with “fat tails”, so that these shocks are relatively likely, the optimal arrangement exhibits maximal segmentation, into components of the smallest size. In contrast, when it is Pareto with “thin tails” the optimum has no segmentation, that is all firms lie in the same component. We can understand the polar outcomes in these two cases as follows: the concern for contagion unambiguously prevails in the first case, that for risk sharing in the second one. The situation is less clear cut when the distribution is given by a mixture of Pareto distributions, one with thin tails and another with fat ones. We show that the optimum features in this case a compromise between the two concerns, as we have a segmentation of the system into identical components of some *intermediate size* (still with uniform internal exposure).

When instead the shock follows a mixture between a Pareto distribution with thin tails and a distribution with positive mass on some large values, we show that modulating the level of exposure to the firms in the system proves a more effective instrument to limit contagion than segmentation (which introduces more rigid circuit breaks in the system, either full or no connection between any two firms). The optimal structure

¹A component of the network is defined as a maximal set of firms that are directly or indirectly connected.

features in fact a single component with, under suitable circumstances, a high and a low levels of exposures between firms.

In a second part of the paper we examine whether the incentives of individual firms to establish linkages with other firms (so as to minimize individual default probabilities) are aligned with social optimality. To this end we consider a simple network-formation game where each firm proposes the formation of a linkage to other firms. For all proposals that are accepted links are formed. The game focuses on the formation of the connected components of the system. We then show that for those scenarios where social optimality requires polar segmentation (i.e. components of maximum or minimum size), equilibrium and socially optimal outcomes coincide. This is not so, however, when optimality prescribes intermediate segmentation. For, in this case, a conflict typically arises: social optimality prescribes components all of the same size, in contrast the equilibrium is asymmetric, with some components of larger size and one, smaller component. While the formation of larger components is individually optimal for all firms that belong to them, the feasibility constraint imposed by the set of firms present in the system forces the remaining firms to lie in an inefficiently small component, where default probabilities are high.

The final part of the paper extends the analysis to the case where firms are *ex-ante* asymmetric. In particular we allow for two kinds of heterogeneity: in the distribution of the shocks and in size. Interestingly, the implications are somewhat different in the two cases. When the heterogeneity concerns the risk characteristics of the shocks hitting the projects of the various firms, optimality requires perfect assortativity, i.e. segmentation along firms' type, with components formed by homogeneous firms, with the same risk profiles of shocks. On the other hand, when firms only differ in their size, and not risk, an irrelevance result holds: the composition of components does not matter for firms' default probability, as long as the exposure of any firm to large and small firms can be properly modulated to "offset" the effect of size. Hence any other, even small advantage, besides risk sharing, of combining large and small firms will optimally lead to the formation of components where firms differ in their size, as in the case of core-periphery networks.

Our model, although stylized, allows for the guidance of concrete policy advice,² in particular concerning how actual policy should deal with the issue of systemic risk. In view of the increasing trend of financial firms' interconnections (see Diebold and Yilmaz 2009), it is now widely judged to be insufficient to rely primarily on the usual approach to the problem, i.e. the regulation of capital requirements.³ Our paper

²This is particularly true in light of the robustness of many of our implications found by Loepte *et al.* (2013). We postpone a summary of the robustness exercise conducted there to the concluding Section 6, once the analysis of the present paper is complete.

³The Basel III agreements require higher overall equity requirements than previous rounds plus an additional capital conservation buffer of 2.5% during expansions to combat the pro-cyclicality of credit.

makes it clear that the derivation of an optimal structure goes far beyond understanding which are the most systemic institutions,⁴ because even in a model with symmetric firms, the structure of the network plays a very important role for the stability of the system. Our paper shows the importance of the distribution of shocks for understanding what is the optimal structure of the network: it provides some rather general results for realistic distributions, and it also provides a tractable framework to study even more complicated problems. We believe this is an important contribution since, as DTTC (2015) points out: “one of the challenges of designing direct measures is the uncertainty surrounding the level of interconnectedness that is deemed undesirable,” and our model (and further extensions) can give reasonably precise answers to that challenge.

Another important aspect we have highlighted is that when different institutions are hit by shocks coming from different distributions, they should belong to separate components. This result adds a novel justification for the separation of investment and commercial banking which was enforced by the Glass-Steagall act from 1933 to 1999.⁵ In our model, the separation may prove beneficial as a way to cope with shocks with different risk profiles hitting alternative types of institutions. It is conceivable that, for this reason, commercial banks should have a much larger component size than investment banks. In fact, several current policy initiatives aim precisely at limiting risk exposures by separating banking activities that have intrinsically different risk levels (the Volcker rule in the U.S., the Vickers and the Liikanen proposals, see DTCC (2015)).

Related literature

The research on financial contagion and systemic risk⁶ is quite diverse and also fast-growing. Hence we shall provide here only a brief summary of some of the more closely related papers⁷.

Allen and Gale (2000) pioneered the study of the stability of interconnected financial systems. They analyze a model in the Diamond and Dybvig (1983) tradition, where a network structure with a single, completely connected component is *always* optimal, i.e. the one that minimizes the extent of default. Our model, in contrast, shows that a richer shock structure can generate a genuine trade-off between risk-sharing

⁴Even when network characteristics have entered the policy discussion, they have done so by focusing attention on institutions that are “too big to fail” – or in the more enlightened versions, “too central to fail,” as in Battiston et al. (2012).

⁵The traditional view was that the separation is necessary to protect consumers from conflict of interest between these institutions, but Kroszner and Rajan (1994), for example, find little evidence for this argument.

⁶There is also a large body of literature that studies the general problem of risk sharing in non financial contexts, largely motivated by its application to consumption sharing in poor economies that lack formal insurance mechanisms. Paradigmatic examples are the papers by Bramoullé and Kranton (2007), Bloch *et al.* (2008), and Ambrus *et al.* (2011).

⁷The reader is referred to Allen and Babus (2009) and to Cabrales *et al.* (2016) for recent surveys of the literature on financial contagion in networks.

and contagion, and that both segmentation and lower density of connections may be optimal.⁸

Three more recent related papers are Elliott *et al.* (2014), Acemoglu *et al.* (2015), and Glasserman and Young (2015). As explained more in detail in Remark 1 in the next section, the nature and form of the financial linkages among firms considered in these papers are partly different, as they may concern not only the asset side but also the liability side of the firms' balance sheet, and entail the presence of some amplification mechanism of the shocks hitting a firm. Another difference with our paper concerns the main focus of the analysis. While the aim of Elliott *et al.* (2014) is to characterize conditions on the structure of the network under which default cascades occur, the primary objective of ours is to characterize the optimal financial structures in diverse scenarios and its relationship with individual incentives. Acemoglu *et al.* (2015) shares with the present one its concern with the optimality of financial networks, but focuses attention on shock distributions that are degenerate Dirac measures concentrated on a given shock magnitude.⁹

An important contribution of our paper that distinguishes it from the papers just mentioned, as well as from others in this literature, is the added generality of our model, in terms of the richness of the set of shock distributions we allow and for which we characterize optimal structures within a large set of financial architectures. This in turn leads to quite diverse optimal networks within the same formal structure (from very extreme structures to more balanced ones, in terms of both segmentation and internal density of connections). We can then see what is the optimal form taken by circuit breaks in the system to limit contagion, while allowing risk sharing, for various shock distributions, as well as the optimal composition of components in the presence of heterogeneity of firms. As we mentioned earlier one of the main obstacles for policy “is the uncertainty surrounding the level of interconnectedness that is deemed undesirable,” (DTCC 2015). Thanks to the relatively simpler modeling framework we examine, we are able to provide results that are both robust and intuitive to lower that uncertainty, as well as a tool of analysis that is easier to generalize even further.

There is also a complementary line of literature that, in contrast with the papers just mentioned, studies the issue of contagion and systemic risk in the context of large networks. While most papers follow a numerical approach, based on large scale simulations¹⁰, Blume *et al.* (2011) use the mathematical theory of random networks to study the optimal average degree of the network and its consistency with individual incentives to create or destroy links.

⁸Babus (2013) uses a similar model but allows for the endogenous formation of links between banks, and shows that banks manage to form networks that are usually resilient to the propagation of shocks.

⁹Glasserman and Young (2015) shares some features with Acemoglu *et al.* (2015), but they allow for continuous distributions of the shocks (as we also do). Their main result is a characterization of the parameter values and networks for which the probability of default of a firm is lower when a shock hits some its neighbors than when the firm is directly hit.

¹⁰See Leitner (2005) and Gai *et al.* (2011).

Another segment of the literature then focuses on the effects on contagion of imperfect information about the shocks hitting the system. For instance, Allen *et al.* (2011) explore the effects on segmented and unsegmented structures of the arrival of a signal indicating that *some* firm in the system will have to default.¹¹

Finally, we should mention the large empirical and policy-oriented literature whose main objective has been to devise summary measures of the network of inter-firm (mostly banks) relationships so as to predict the likelihood of systemic failures. For example, Battiston *et al.* (2012) or Denbee *et al.* (2011) propose measures of centrality in networks. Of particular interest in this respect is Elsinger, Lehar and Summer (2011) who, using Austrian data, show that correlation in banks' asset portfolios is the main source of systemic risk.¹²

The rest of the paper is organized as follows. Section 2 describes the economic environment and the possible patterns of linkages among firms. Section 3 characterizes the optimal financial structures for various properties of the shock distribution. Section 4 addresses the issue of network formation and the relationship between individual incentives to form linkages and social optimality. Section 5 extends the analysis to allow for heterogeneity in firms' size and in the distribution of the shocks that may hit firms. Section 6 concludes. For convenience, the proofs of our results are relegated to the Appendix.

2 The Model

2.1 The Environment

Our benchmark scenario considers an environment with N ex ante identical financial firms (say, banks). We assume N is even. Each firm i has one project which yields a gross return given by a random variable \tilde{R}_i as follows. With probability $1 - q$ the return equals its 'normal' value R , while with probability q the firm's investment is hit by a shock and its return equals $R - \tilde{L}_i$, where \tilde{L}_i is a positive valued random variable with the same distribution for all i . Every firm has then total liabilities towards outside investors (e.g. deposit or bond holders) equal to M .¹³

¹¹On a similar line, Caballero and Simsek (2013) and Alvarez and Barlevy (2014) study how informational contagion can amplify the contagion resulting from mutual exposure between firms.

¹²A related emphasis on cross-ownership and investment has been pursued as well by the literature on "balance sheet effects" to understand the Asian financial turmoil in the late 90's (Krugman 1999) as well as the current crisis (Ahrend and Goujard 2011). These latter approaches are very much in line with our model, which precisely highlights portfolio correlation as the key driver of default risk.

¹³We can think of these liabilities as issued to raise external funds to finance the firm's project. See Cabrales et al. (2014) for the analysis of the case where M is endogenously determined on the basis of investors' preferences and firms' default

We model financial linkages by allowing each firm to hold claims to the returns obtained from the projects of other firms, together with a fraction of the claims on its own project. More precisely, for any $i, j = 1, \dots, N$, let us denote by $a_{ij} \geq 0$ the fraction held by firm i of the outstanding amount of claims to the yield of the project of firm j . Hence the pattern of asset holdings across the N firms in the economy can be described by a matrix A of the following form:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{pmatrix}. \quad (1)$$

The return on the *assets owned* by any given firm i is then a weighted average of the yield of the firm's own project and the yields of the projects of the other firms. Subtracting from this the value of the firm's liabilities, given by M , we obtain the following expression for the net financial position of firm i :

$$\sum_{j=1}^N a_{ij} \tilde{R}_j - M. \quad (2)$$

Hence inter-firm linkages arise through the asset side of the firms' balance sheet, since the value of the assets of firm i may be related in general to the value of the assets of other firms $j \neq i$. In contrast, note that no linkage is present on the liability side.

When the realization of the return on the assets held by firm i falls short of M , that is the expression in (2) has a negative sign, the firm *defaults*. Default is costly: in addition to liquidation costs there are opportunity costs deriving from the fact that a defaulting firm stops operating, and hence it loses any future earnings possibility.

2.2 Shock structure

We specify now some properties of the probability distribution of the shock \tilde{L}_i that may hit the yield of the project of firm i . Conditionally on such shock hitting firm i , with probability π the shock is 'small' (labeled s) and the project experiences a loss of size \underline{L} . With probability $1 - \pi$ the shock is 'big' (labeled b), and is described by a random variable \tilde{l} , with support¹⁴ $[\underline{L}, \infty)$ and cumulative distribution function $\Phi(l)$.

probabilities.

¹⁴We allow the support of \tilde{l} to go to ∞ purely for technical convenience. The analysis could have been equivalently carried out by truncating the distribution at some upper bound $\bar{L} \leq R$ (so as not to violate limited liability), with $\bar{L} > N(R - M)$,

Summarizing, the gross return on the project of an arbitrary firm i is:

$$\widetilde{R}_i = \begin{cases} R & \text{with prob. } 1 - q \\ R - \underline{L} & \text{with prob. } q\pi \\ R - \widetilde{l} & \text{with prob. } q(1 - \pi) \end{cases} \quad (3)$$

For the risk of default to be an issue, we assume:

- A1.** (i) $R(1 - q) > M$,
(ii) $R - \underline{L} < M$.

The first inequality ensures that a firm's project is viable, that is, its expected return exceeds what must be paid to lenders. On the other hand, the second inequality implies (since $\widetilde{l} \geq \underline{L}$) that if a firm can only draw on the revenue generated by its project, it is surely unable to pay depositors (and hence must default) whenever a shock, *whether small or large* hits its return.

We assume that shocks are rare and thus at most one project is hit by a shock. This can be motivated by postulating that, even if shocks hit projects in a stochastically independent manner, the probability q that a shock hits any given one is so low that the probability that two or more shocks arrive in a single period is of an order of magnitude that can be ignored.¹⁵

On the nature of those shocks, we make the following key assumption:

- A2.** (i) $\pi > N(1 - \pi)$,
(ii) $\frac{1}{2}(R - \underline{L}) + \frac{1}{2}R \geq M$.

Part (i) of the above assumption says that s (small) shocks are significantly more likely than b (big) ones. In particular, it is more likely that a given firm is hit directly by an s shock than by a b shock. Part (ii) ensures, for reasons that will become clearer and more precise in the next section, that the presence of sufficiently strong linkages to other firms allows a firm to fully insure against the s shocks, i.e. it guarantees that the firm never defaults when an s shock hits a firm in the system.

so that all the resources in the system are not sufficient to absorb this extreme realization of the shock and avoid any firm defaulting.

¹⁵Or, as an extreme formalization of this idea, we could model time continuously and assume that the arrivals of small and big shocks to each firm are governed by independent Poisson processes with fixed rates π and $1 - \pi$, respectively. Then, the probability that two shocks arrive simultaneously is zero.

2.3 Interfirm Linkages and Financial Structures

The matrix A describes the pattern of financial linkages among the N firms. As we see from (2), when $a_{ij} > 0$, a firm i is exposed to the shocks hitting the return on firm j 's project. The elements of A specify not only which firm has a financial linkage to any given firm i , but also the strength of this linkage. Formally, we can think of A as the adjacency matrix of a weighted directed network, with a_{ij} representing the intensity of the link from firm j to firm i . To fix ideas, it may help to conceive the reciprocal exposure between two firms as the result of exchanges of assets among the firms. On this line, we can equivalently interpret the linkages with higher intensity as describing direct linkages and those with lower intensity as describing indirect linkages, that is, relate intensity to distance in an underlying network that specifies the trade possibilities among firms.¹⁶

The elements of A can take any value between 0 and 1. To ensure some balance among the positions of different firms, it is natural to require that not only the columns but also the rows¹⁷ of A sum to 1:

$$\sum_{j=1}^N a_{ij} = \sum_{i=1}^N a_{ij} = 1 \quad i, j = 1, 2, \dots, N. \quad (4)$$

The presence of financial linkages allows a firm to reduce its exposure to the shocks that may hit the return on its own project and hence lower the probability that a firm defaults when its project is hit by a shock. At the same time these linkages imply that the firm is now exposed also to the shocks hitting firms with which the firm is connected. This may generate a sort of contagion: a shock hitting the project of any firm may affect many other firms, decreasing the value of the assets they hold and, possibly, may generate widespread default.

More precisely, when a shock of size L hits the return on the project of firm i , this firm defaults if

$$a_{ii}(R - L) + \sum_{j \neq i}^N a_{ij}R < M. \quad (5)$$

¹⁶See Cabrales et al. (2014) for a formal derivation of possible patterns of linkages arising from the iteration of exchanges of assets among firms located in a network. In this analysis, letting B be a matrix describing the pattern of exchanges of assets among firms that are directly connected among them, A is then given by B^T for some $T > 1$ describing the number of iterations of exchanges of assets among neighboring firms (which can be interpreted as rounds of securitization and trade).

¹⁷A condition of this kind would be needed if we think of the firms' cross-holdings as resulting from exchanges of assets among them.

When such a shock hits the return on the project of some other firm $k \neq i$, firm i defaults if

$$a_{ik}(R - L) + \left(a_{ii} + \sum_{j \neq i, k}^N a_{ij} \right) R < M. \quad (6)$$

Hence the lower a_{ii} , the less likely expression (5) is satisfied. At the same time, given (4), this also means that it is more likely that (6) holds. In the environment considered, there is thus a *trade-off between risk-sharing and contagion*.

REMARK 1 *In our model the linkages among firms capture the fact that they are exposed to common shocks, given their mutual ownership of claims on the yields of the different projects. An important consequence of this is that the default of one firm has, per se, no direct implication on the solvency of other firms. Therefore, the possibility of contagion from a large shock hitting one firm only comes from the correlation of the returns on the assets held by firms.*¹⁸

The specifications in Acemoglu al. (2015) and Glasserman and Young (2015) differ from ours in that financial linkages among firms are due to mutual borrowing and lending relationships. They concern, therefore, both the asset and liability side of the firms' balance sheet. On the other hand, in Elliott et al. (2014) linkages only affect the asset side, as in our case, but are a result of the cross-ownership of equity instead of commonly held assets. In all these papers, in contrast to ours, it is the default of the firm hit by a shock what produces (and, in Elliott et al. (2014), amplifies, because of the fixed costs of bankruptcy) contagion. In spite of the differences in the microfoundations of the linkages among firms, as shown in Cabrales et al. (2016), in all the situations considered in these papers the net worth of each firm can be written as a function of the realization of the returns on the projects of the firms in the system, as in ours (and the net worth is all what matters to determine firms' default). The main difference is that in the papers recalled above this function is non linear, and discontinuous, while it is linear in our set-up. An important point noted by Cabrales et al. (2016) is that the effects of the pattern of linkages among firms (as described by the matrix A) on contagion, and more generally on the way in which shocks propagates through the system do not depend in any fundamental way on this difference. The current specification is clearly more tractable and allows so to reach more general conclusions that do not depend on the specific mechanism which generates contagion.

The set of possible financial structures, as described by the matrix A , is quite large. One of the primary objectives of this paper is to compare the performance of different financial structures in the face of various

¹⁸Lazear (2011) in a widely discussed Wall Street Journal article, refers to this as the 'popcorn' mechanism. He contrasts it to the 'domino' effect, where it is the *default* of one firm that triggers the contagion to other firms.

properties of the probability distribution $\Phi(\cdot)$ of the large shocks that may hit the system. Performance will be assessed in terms of the expected number of firms' defaulting. As explained in the next section, when investors are risk neutral this amounts to maximizing net surplus. To help organize our ensuing discussion it is useful to identify three important dimensions that distinguish the different financial networks.

The first dimension is the degree of *externalization of risks*, i.e. the strength of linkages to other firms. This is captured by the relative value of the off-diagonal terms of A with respect to its diagonal terms. The lower the value of the diagonal terms, $(a_{ii})_{i \in N}$, the more risk is externalized but, of course, the stronger is the force of contagion as well. In the extreme case where $A = I$, firms do not have any financial linkage to other firms. Then every firm is in isolation, neither shares nor spreads any risk, and hence it has to bear the whole magnitude of a shock hitting its own project.

A second relevant dimension is *segmentation*, which describes the extent to which firms are connected (directly or indirectly) to many or only a few other firms, that is, if and how the system is split into disjoint components.¹⁹ At one end we have the case of no segmentation, where the system consists of a single component: each firm is then connected - with limited exposure - to every other firm. At the other end, we have maximal segmentation where the system is divided into several components of small size and each firm is only connected²⁰ - with high exposure - to one other firm.

Finally, a third dimension of interest has to do with the *density of the connections* within each component. One natural possibility is to have every firm connected with the same intensity, which we can interpret as being directly connected, to every other firm in its component. For that reason in this case there is uniform exposure to each other's risk. But we can also have any other possible pattern of connections, with varying intensity, within components.

2.4 Payoffs and Optimality

In the environment described, the financial structure determines how shocks hitting any firm are distributed over all the system of firms. As long as no firm defaults, this only entails a redistribution of resources across firms and between creditors and owners of the firms. But when defaults occur, since these are costly, the amount of available resources is also negatively affected, the more so the larger the number of firms defaulting. When all investors are risk neutral we can assess welfare in terms of social surplus, given by the sum of the revenue generated by the firms' assets, less the costs of default. Since these costs are the same

¹⁹As a follow-up on footnote 1, in the present environment where the network is a weighted one, we define a component as a minimal subset of firms with the property that no firm in this subset has a linkage to firms outside it.

²⁰In these two cases we take as given the degree of externalization of risks.

for all firms, we can then say that welfare, as measured by total surplus, is maximal when the number of firms defaulting is minimal.²¹ Also, any firm maximizes its value, in the presence of risk neutral investors, when it minimizes its probability of default.

Assumptions A1(ii) and A2(i) imply that a firm with no linkages always defaults when hit by an s shock, and that s shocks are much more likely than b shocks. It then follows from A2(ii) that the probability of default of an arbitrary firm i is always lower when its degree of externalization, a_{ii} , is at least equal to one half.²² The presence of linkages to other firms satisfying this property is always beneficial, since it allows a firm to fully insure against the s shocks. Given our aim to characterize optimal financial structures, we shall therefore focus our attention in what follows on the set of structures that display a common degree of externalization of risks: $a_{ii} = \alpha$ for all i , for some α close to $1/2$.²³ This ensures that no firm defaults when an s shock hits.²⁴ All the financial structures we consider have then the same diagonal terms, but they can differ widely in terms of the values of the off-diagonal terms. That is, we impose no restriction in terms of the two other aforementioned dimensions, allowing for all possible degrees of segmentation and of density of connections. An important implication is that financial structures differ for the ability firms have to withstand b shocks when they hit some other firm in the system. This allows us to focus our attention on their effects in terms of limiting contagion. While (5) takes the same value for all the structures, and so the probability that a firm defaults when directly hit by a shock is the same, the validity of (6) depends in fact on the value of the off diagonal terms of A , a_{ik} .

Concerning segmentation, this can be measured by the number C of components and the size of each of them, that is the number of firms in each component. Let K_i denote the number of firms to whom firm i is linked (directly or indirectly). Since in the benchmark model all firms are ex ante identical, in the analysis here we shall consider components where K_i takes the same value for each firm i in a given component.²⁵ The size of the component is then given by $K_i + 1$, that is, every firm has a linkage to every firm lying in its component (and no linkage to firms lying outside it). But we shall still allow for asymmetries, as different

²¹Beale et al. (2011) propose a similar criterion to evaluate and compare different financial systems, based on the minimization of a “systemic cost function” defined as the expectation of a convex function of the number of defaults in the system. More generally, if we drop the risk neutrality of investors, not only the expected number of defaults matters, but also its variability.

²²More specifically we require α to be such that $\alpha(R - \underline{L}) + (1 - \alpha)R \geq M$.

²³A further optimization with respect to the level of α (subject to the constraint in footnote 22) would not yield extra insights with respect to the issues we are mostly concerned, given our primary focus on contagion and the probability of default when indirectly hit by shocks.

²⁴While this risk sharing target imposes an upper bound on α , other considerations, in terms of classical moral-hazard arguments, suggest a lower bound on α . This is the concern reflected, for example, by a well-known provision in the recent Dodd-Frank act passed in the USA to strengthen the regulation of the financial system. By virtue of this new legislation, under certain circumstances “a securitizer is required to retain not less than 5 percent of the credit risk...” (see <http://www.sec.gov/about/laws/wallstreetreform-cpa.pdf>).

²⁵See Section 5 for an analysis of the case where we allow for asymmetries among firms and hence also where K_i may vary across firms within a component.

components may be of different size and exhibit different density of connections. In the extreme cases of no segmentation and maximal segmentation we have, respectively, $K_i = N - 1$ and $K_i = 1$ for all i .

Rows and columns of A can always be rearranged so that this matrix has a block diagonal structure, with the blocks corresponding to the network components. The density of the connections within any component (say, of size $K + 1$) is then formally described by the entries off the main diagonal of the submatrix A_K , that is the $K + 1$ -dimensional adjacency matrix that includes only the entries corresponding to the $K + 1$ firms in the component. In the case of uniform exposure (that is, maximal density) we have:

$$A_K = \begin{pmatrix} \alpha & (1 - \alpha)/K & \cdots & (1 - \alpha)/K \\ (1 - \alpha)/K & \alpha & \cdots & (1 - \alpha)/K \\ \vdots & \vdots & \ddots & \vdots \\ (1 - \alpha)/K & (1 - \alpha)/K & \cdots & \alpha \end{pmatrix}, \quad (7)$$

One example of an alternative possibility is a situation where exposure monotonically decreases with distance from the diagonal, so every firm is directly connected only with two other firms and indirectly, with varying distance, with all other firms in the component:

$$A_K = [a_{ij}]_{i,j=1}^{K+1} \text{ such that: } a_{ij} > a_{iq} \text{ whenever } |i - j| < |i - q|$$

With no segmentation and maximal density we have $a_{ij} = (1 - \alpha)/(N - 1)$ for all i, j , so that every firm is equally exposed to every other firm in the system. In this situation we have maximal risk sharing, but also maximal possibility of contagion. A first way in which the spread of contagion can be limited is via segmentation, so that $a_{ij} = 0$ for some j . But this also means that, whenever $a_{ij} > 0$ its value is on average greater than $(1 - \alpha)/(N - 1)$, the more so, the greater is segmentation. Hence the exposure of a firm is more concentrated on a subset of firms, that is risk sharing is also lower. A lower density of connections provides another way in which contagion can be limited: in that case too, the exposure is unevenly distributed, but now this happens by modulating the intensity of linkages within a component, rather than by dividing the system into disjoint components.

3 Optimal Financial Structures

In this section we will determine the optimal financial structure, defined by the degree of segmentation and the intensity of linkages within each component that allows to minimize the expected number of defaults in the system, for different properties of the cumulative distribution function $\Phi(\cdot)$ of the b shocks that may hit firms. The set of admissible financial structures is quite large, also no restriction has been imposed on the probability distribution. To state formally the optimization problem yielding optimal financial structures we proceed then as follows.

From (6) we see that, when a shock hits some firm k , firm $i \neq k$ defaults if the (random) magnitude of this shock is sufficiently large relative to the size of the exposure of i to k , that is

$$L > \frac{R - M}{a_{ik}}.$$

Hence the probability that i defaults, conditionally on the event that firm k is hit by a b shock, is:

$$1 - \Phi((R - M)/a_{ik}). \quad (8)$$

In what follows, we will use the notation $F(a_{ik})$ to indicate the above expression in (8), written as a function of the choice variable a_{ik} .

To formulate the social optimization problem we proceed in two steps. First, we find the optimal pattern of connections in a component of size $K + 1$, for any $K = 1, \dots, N - 1$, that is the one minimizing expected defaults in a component. Using the fact that a b shock can hit any of the firms with equal probability, this is obtained as a solution of the following problem:

$$[\mathbf{P}(K)] \quad V(K) = \min_{[(a_{ij})_{j \neq i}]_{i=1}^{K+1}} \frac{1}{K+1} \sum_{i=1}^{K+1} \sum_{j \neq i} F(a_{ij}) \quad \text{s. t.} \quad (9)$$

$$\begin{aligned} \sum_{j \neq i}^{K+1} a_{ij} &= \sum_{i \neq j}^{K+1} a_{ij} = 1 - \alpha, \quad \forall i \neq j \\ a_{ij} &\geq \varepsilon \quad \forall i \neq j \end{aligned} \quad (10)$$

for some sufficiently small $\varepsilon > 0$. The expression appearing in the objective function specifies the expected fraction of firms defaulting in a component of size $K + 1$ when indirectly hit by a b shock, conditionally on such a shock hitting a firm in the system. To minimize this expression amounts to minimizing the expected

number of defaults in the component since, as we noticed, the probability that a firm defaults when directly hit by a b shock is the same across all admissible financial structures. Constraints (9) and (10) of problem $[\mathbf{P}(K)]$ restate, respectively the balance condition (4) and the condition that, within the component, every firm is linked to every other firm lying in it.²⁶ We denote then by $V(K)$ the value of the objective function at a solution of problem $[\mathbf{P}(K)]$, constituting the ‘optimal value of a component of size $K + 1$ ’.

In the second step we find the optimal segmentation, that is, the optimal partition of the N firms into connected components:

$$[\mathbf{SP}] \quad \min_{K_c \in \{1, \dots, N-1\}, C \in \{1, \dots, N/2\}} \sum_{c=1}^C V(K_c)(K_c + 1) \quad \text{s. t.} \\ \sum_{c=1}^C (K_c + 1) = N$$

where the expression appearing in the objective function gives the expected number of firms in the system that default when indirectly hit by a b shock (again, conditionally on such a shock hitting a firm in the system).

To derive the solution of problem $[\mathbf{SP}]$, it will often be convenient to consider the following *individual optimization problem*:

$$[\mathbf{IP}] \quad \min_{(a_{ij})_{j \neq i}} \sum_{j \neq i} F(a_{ij}) \quad \text{s. t.} \\ \sum_{j \neq i}^N a_{ij} = 1 - \alpha \tag{11}$$

$$a_{ij} \geq 0, \quad \forall j \neq i \tag{12}$$

This problem consists in the choice of the pattern of exposures of firm i to other firms in the system - that is, of the off diagonal elements of the i -th row of matrix A - that minimizes the probability of default of firm i when indirectly hit by a b shock. As argued in Section 2.4, the value of firm i is maximal when the firm’s probability of default is minimal. Problem $[\mathbf{IP}]$ will then also play a role in the next section when we will investigate, after having studied the social optimum, the individual firms’ incentives to establish linkages.

The key difference between $[\mathbf{SP}]$ and $[\mathbf{IP}]$ lies in the fact that in the second optimization problem each individual firm i chooses the intensity of its linkages unilaterally. In so doing, however, the firm abstracts from the constraints that relate such values to those of other firms. These are the constraints that appear in

²⁶To ensure the closedness of the constraint set of problem $[\mathbf{P}(K)]$ we state the last condition by imposing a strictly positive, though arbitrarily small, lower bound on the admissible values of a_{ij} .

the second condition of (9) and in (10), capturing the fact that linkages are formed via trades among firms and, in order to carry out any desired asset exchange, the two parties must agree to it. These constraints are ignored here, hence the values obtained by solving **[IP]** for each firm i may not be feasible. When they are feasible, that is when the values of a_{ij} solving **[IP]** satisfy the missing constraints, these values also constitute a solution of the social optimum problem **[SP]**. The objective functions of the two problems are in fact consistent: to minimize the probability that any single firm defaults ensures that the expected number of defaults in the system is also minimized.²⁷

As we will see in what follows, it happens in several cases that at a solution of **[IP]** the missing constraints are satisfied, so that a social optimum can be found by solving the simpler problem **[IP]**. This is not always true, though: when this does not happen, not only a solution of **[SP]** cannot be obtained, but also there may be a gap between the individual incentives to form linkages and the socially optimum pattern of linkages (though in both cases, as we noticed, the objective is to minimize defaults).

Having stated formally the social optimization problem, we proceed to characterize optimal financial structures under a diverse range of shock scenarios. We begin by deriving some clear-cut conclusions in the cases where the function $F(a_{ij})$ is either uniformly convex or uniformly concave. As we shall show in Remark 2 below, this convexity/concavity property turns out to be a straightforward generalization of the “fat tails/thin tails” property that often arises in the context of Pareto (or power-law) distributions.

PROPOSITION 1 *If the function $F(\cdot)$ is strictly convex, a solution $[(a_{ij}^*)_{j \neq i}]_{i=1}^N$ of problem **[SP]** is given by*

$$a_{ij}^* = \frac{1 - \alpha}{N - 1} \quad i = 1, 2, \dots, N; \quad j \neq i. \quad (13)$$

that is, there is a unique component containing all firms with uniform exposure.

If instead $F(\cdot)$ is strictly concave, a solution $[(a_{ij}^)_{j \neq i}]_{i=1}^N$ of problem **[SP]** is characterized by a one-to-one mapping $m : \{1, 2, \dots, N\} \rightarrow \{1, 2, \dots, N\}$ with $m(k) \neq k$ and $m^2(k) = k$ for all k such that, for each $i = 1, 2, \dots, N$,*

$$a_{im(i)}^* = 1 - \alpha \quad \text{and} \quad a_{ij}^* = 0 \quad \text{for all } j \neq m(i), \quad (14)$$

that is, there are $N/2$ components each of size 2.

Proof. See the Appendix.

²⁷Formally, if a_{ij} for $i \neq j$ is such that $\sum_{j \neq i} F(a_{ij})$ is minimal for all i clearly ensures that $\sum_{i=1}^N \sum_{j \neq i} F(a_{ij})$ is also minimal.

The previous result is intuitive. As we said, the key consideration underlying the optimal pattern of exposures of a firm is the trade-off between risk sharing and contagion. In the clear-cut scenario where the default probability of a firm is given by a function of the exposure to another firm displaying a uniform curvature, whether the benefits of risk sharing or the risks of contagion prevail is captured by this curvature. If the stated function is convex, the marginal effect on the default probability of increasing exposure is relatively low at low levels of exposure. Hence it is optimal to spread out a firm's exposure as much as possible among all the other firms and thus the optimal arrangement is to have a single component with uniform exposure. The opposite considerations apply when the function in question is concave, in which case the firm minimizes its default probability by restricting the sources of contagion so as to minimize the risk of contagion, i.e. by choosing components of minimum size.

As anticipated, we explain in the following remark that the uniform curvature of the function $F(\cdot)$ postulated in our previous result can be understood as a generalization of a thin/fat tails condition, which is particularly easy to formulate if the shocks are Pareto distributed.

REMARK 2 Suppose the b shock \tilde{l} is Pareto distributed on $[\underline{L}, \infty)$ with continuous density

$$\phi(l) = \gamma \frac{\underline{L}^\gamma}{l^{\gamma+1}}. \quad (15)$$

Straightforward computations then show that, for any $a_{ij} \in (0, 1]$, we have:

$$\begin{aligned} F(a_{ij}) &= 1 - \int_{\underline{L}}^{(R-M)/a_{ij}} \phi(l) dl \\ &= 1 - \gamma \underline{L}^\gamma \int_{\underline{L}}^{(R-M)/a_{ij}} \frac{1}{l^{\gamma+1}} dl = 1 + \underline{L}^\gamma \left(\left(\frac{a_{ij}}{R-M} \right)^\gamma - \frac{1}{\underline{L}^\gamma} \right) \\ &= a_{ij}^\gamma \left(\frac{\underline{L}}{R-M} \right)^\gamma \end{aligned}$$

Therefore if $\gamma < 1$, and hence the Pareto distribution displays fat tails (in the sense of having unbounded second-order moments), the induced function $F(\cdot)$ is uniformly concave. Alternatively if $\gamma > 1$, and thus the Pareto distribution has thin tails (i.e. the second-order moments are bounded), the function $F(\cdot)$ is convex.

In view of the above remark and Proposition 1, the following corollary readily follows.

COROLLARY 1 Suppose the b shock is Pareto distributed with density given by (15), and let $[(a_{ij}^*)_{j \neq i}]_{i=1}^N$ be the socially optimal matrix of exposures. Then, its entries are given by (14) if $\gamma < 1$, and by (13) if $\gamma > 1$.

To sum up, if shocks follow a Pareto distribution, only two polar types of structures can be optimal: either maximum segmentation into the smallest components, or no segmentation at all. Which of these two possibilities arises hinges solely upon whether shocks large enough to exceed the capacity of the system to absorb them without firms defaulting are relatively frequent or not. With thin tails these large shocks are not very likely, hence the predominant consideration is to enhance risk sharing rather than to control contagion: therefore default is minimized by having all firms in a single component with uniform exposure. On the other hand, with fat tails the main concern becomes limiting the spread of contagion, which is optimally achieved by breaking the system into disjoint components of minimal size. Thus, all the adjustment to the increase in risk is achieved in this case by maximally increasing segmentation, not by varying the intensity of linkages within components.

Such a clear-cut and polarized conclusion is useful to highlight the essential forces at work. However, as we will see in the analysis that follows, when richer, more complex sets of shock distributions are considered, the choice of intermediate degrees of segmentation and/or of modulating the intensity of linkages within connections may also obtain at the optimum. To gain a clear understanding of the best way to address the trade-off between risk sharing and contagion in richer environments, it is convenient to remain within the realm of all Pareto distributions but convexify the scenario by allowing mixtures of them. In this generalized context, we now show that simple mixtures of fat-tail and thin-tail distributions may induce optimal configurations that display intermediate levels of segmentation.

PROPOSITION 2 *Suppose the distribution of \tilde{l} is given by a mixture of Pareto distributions on $[\underline{L}, \infty)$, with continuous density $\phi(l) = p\gamma_1 \frac{\underline{L}^{\gamma_1}}{l^{\gamma_1+1}} + (1-p)\gamma_2 \frac{\underline{L}^{\gamma_2}}{l^{\gamma_2+1}}$, $0 < p < 1$ and $\gamma_1 > 1 > \gamma_2$. Assume:*

$$\hat{K} \equiv \frac{1 - \alpha}{\left[\frac{(1-p)(1-\gamma_2)}{p(\gamma_1-1)} \right]^{\frac{1}{\gamma_1-\gamma_2}} \left(\frac{R-M}{\underline{L}} \right)} \in (1, N-1).$$

Then the solution of problem [IP] has \hat{K} nonzero terms²⁸ a_{ij} , all equal to $\hat{a} \equiv \left(\frac{(1-p)(1-\gamma_2)}{p(\gamma_1-1)} \right)^{\frac{1}{\gamma_1-\gamma_2}} \left(\frac{R-M}{\underline{L}} \right)$ and the remaining $N - 1 - \hat{K}$ ones all equal to 0.

Proof. See the Appendix.

The above result shows that in this case the solution of the individual problem [IP] features for each firm a positive level of exposure \hat{a} to more than one but not all the other firms. Now the function $F(\cdot)$, describing the default probability when a firm is indirectly hit by a shock, is first concave and then convex.

²⁸We ignore here for simplicity the constraint that K should be an integer.

The condition $\hat{K} \in (1, N - 1)$ requires that both the possibilities that the shock distribution have fat and thin tails are attributed significant probability. Hence neither of the two polar forces at work dominates and the trade-off between risk sharing, and contagion no longer has an extreme solution that privileges only one of them. As shown in Proposition 2, the solution of **[IP]** exhibits some risk sharing, with the formation of linkages to more than one firm, but also some concern for contagion, since such linkages are formed only with a subset of firms. Also, the route chosen to limit contagion is only segmentation, not a variation of the intensity of linkages with other firms: the exposure is uniform to all firms with whom a linkage exists.

Another important feature of the solution of problem **[IP]** for each firm i is that in this case the solution does not generally satisfy the additional constraints present in problem **[SP]**, (9) and in (10). It is in fact immediate to see that we cannot select \hat{K} off-diagonal elements to be equal to \hat{a} in each row of the matrix A , with the remaining elements equal to 0, so as to form disjoint components within which every firm is linked to every other firm lying in it, unless $N / (\hat{K} + 1)$ happens to be an integer, and this is typically not the case. Hence to find a social optimum, solving problem **[SP]**, we need now to take explicitly into account in the optimization problem the feasibility constraints connecting the various rows of A . How should the division of the system into components and their internal structure be adjusted so as to take these constraints into account? Is the social optimum asymmetric, with all but one components exhibiting the same pattern of connections as at the solution of problem **[IP]** and the last component taking the slack? Or is it rather a more symmetric configuration where all components adjust their sizes to satisfy the feasibility constraint? We provide the answer and characterize the optimum in the following:

PROPOSITION 3 *Under the same assumptions of Proposition 2, if in addition $\gamma_1 \in [2, 3]$ and $N < 2(\hat{K} + 1)$, a socially optimal matrix of exposures $[(a_{ij}^*)_{j \neq i}]_{i=1}^N$ solving problem **[SP]** is symmetric and such that, in each row $i = 1, 2, \dots, N$, there are $K^* < \hat{K}$ off-diagonal elements equal to $a^* = (1 - \alpha)/K^*$ and the remaining ones equal to 0.*

Proof. See the Appendix.

In the proof we characterize the optimal structure of a component of size $K + 1$, obtained as solution of problem **[P(K)]**. We show that for $K \leq \hat{K}$ the optimum features equal exposure to all other firms in the component, hence the optimal value of a component is $V(K) = KF((1 - \alpha)/K)$. In contrast, for $K > \hat{K}$ the optimal value $V(K)$ is close to $\hat{K}F((1 - \alpha)/\hat{K})$. We also show that for $K \leq \hat{K}$ the value function $V(K)$ is convex. These properties of $V(\cdot)$ imply that at a solution of the social optimum problem **[SP]** we have - at least - two disjoint components, all of the same size $K^* + 1 < \hat{K} + 1$. From the characterization of the

solution of problem $[\mathbf{P}(K)]$ it also follows that within each component at the optimum each firm has the same exposure to any other firm lying in its component. Thus, the need to take the feasibility constraints into account affects all components, as they are all different from the solution of the individual problem $[\mathbf{IP}]$ shown in Proposition 2, and it affects them all in the same way given the symmetry of the optimum.

The difficulty in establishing the result (and the need for some parametric constraints) arises because the mixture of fat and thin-tailed distributions yields a non-convex optimization problem, and the arguments become quite constructive. The main message is then that, in the presence of changes in the curvature of the function describing the default probability, the optimal solution may not be extreme but it entails a combination of risk sharing and circuit-breaks to limit contagion, and also that feasibility considerations constrain the optimal structure of individual components.

To complete the analysis of this section, we present a final result showing that, if the shock scenario under consideration is further enriched to allow for mass points, the optimal financial structure may exhibit quite different properties. In particular, we show that in such a case, the socially optimal way to limit contagion no longer entails segmentation, but rather modulating the internal connectivity within components. More specifically:

PROPOSITION 4 *Suppose the distribution of \tilde{l} is a mixture, with weights p and $1 - p$, of (i) a distribution with continuous density and c.d.f. $\Phi(\cdot)$ inducing a probability of indirect default described by a function $F(\cdot)$, as in (8), that is strictly convex, and (ii) a Dirac measure concentrated on a value L_1 . Assume*

$$L_1 > \frac{(R - M)(N - 1)}{1 - \alpha}, \quad (16)$$

and

$$1 - p + p \left(F \left(\frac{1 - \alpha}{N - 1} \right) - F \left(\frac{R - M}{L_1} \right) \right) - p F' \left(\frac{1 - \alpha}{N - 1} \right) \left(\frac{1 - \alpha}{N - 1} - \frac{R - M}{L_1} \right) > 0. \quad (17)$$

Then the solution $[(a_{ij}^)_{j \neq i}]_{i=1}^N$ of problem $[\mathbf{SP}]$ is such that, for each $i \neq j$, $a_{ij}^* \in \{a', a''\}$, for some $a' > a'' = (R - M)/L_1$.*

Proof. See the Appendix.

The above proposition considers a situation where the shock follows a mixture between a distribution for which the default probability increases with exposure in a convex manner (as in the case of Pareto with thin tails) and another distribution with mass fully concentrated on a single value L_1 . As we saw in Proposition 1, with the first distribution the benefits of risk sharing are predominant, favoring a wide (and hence low-

intensity) exposure to all firms in the system. The mixture however attributes positive probability to the possibility that a large shock hits the system and condition (16) implies that, were firms arranged in a single component with uniform exposure, contagion would be maximal when such a shock hits, as all firms would default. Condition (17) is satisfied if the weight p on the first distribution is sufficiently low, or $F(\cdot)$ is close enough to be a linear function²⁹. As a consequence at the optimal structure we need to limit firms' exposure with respect to the case of a single component with uniform exposure.

Again a suitable compromise between risk sharing and contagion must be found, but in this case – in contrast with the scenario studied in Proposition 3 – it is not achieved by operating on the segmentation dimension but rather by adjusting the intensity of the connections among firms in a component. At the optimal structure every firm is linked to any other firm - hence there is a single component - but with two possible levels of exposure, the lowest one equal to $(R - M)/L_1$. Intuitively, the reason for this alternative way to cope with the threat of contagion has to do with the fact that this lower level of exposure suffices to prevent default from happening when a large shock of size L_1 hits a firm in the system. It is then possible to enjoy the risk sharing benefits of lying in the largest possible component (where relatively small shocks are best tackled), while moderating the effect of the occasional, large shock. It is interesting to observe that a similar result also holds even if we allow for mixtures with distributions putting positive mass on two or more shocks: it can be shown that at the optimum we may still have only two levels of exposure.

4 Equilibrium and optimality

The analysis undertaken in the previous section has characterized the patterns of financial linkages among firms which are optimal from a social welfare viewpoint. An important issue is whether these configurations are consistent with the choices made by individual firms – in other words, whether social and individual incentives are aligned.

To address this issue, we need to model the process of network formation among firms, specifying how individual firms decide to create or destroy links and choose their intensity, thus shaping their pattern of exposures. Such a network formation analysis can be carried out at different levels of detail, and under different assumptions on firms' behavior. Here our choice in these respects is motivated by the following two-fold objective. First, mutual exposures should require the consensus of those involved. We want then to allow firms sufficient discretion and flexibility to establish linkages with any other firm in the system.

²⁹By the convexity of $F(\cdot)$ we have in fact: $\left(F\left(\frac{1-\alpha}{N-1}\right) - F\left(\frac{R-M}{L_1}\right)\right) - F'\left(\frac{1-\alpha}{N-1}\right)\left(\frac{1-\alpha}{N-1} - \frac{R-M}{L_1}\right) < 0$.

Second, we want to model the decision process regarding the intensity of linkages among firms in any connected component in a simple enough manner that the theoretical framework is rendered tractable. This suggests describing such internal process in a reduced form, so that only the formation of the groups of firms constituting the connected components of the system is studied from an explicitly strategic viewpoint.

To define matters formally, we first specify the network formation game and then the equilibrium concept to be applied to it. The proposed game is a variation of the classical “announcement game” used in the network literature (see Myerson (1991)) and consists of two stages:

- (1) Each firm independently submits its proposals concerning the set of other firms to which it wants to connect. Formally, a strategy of each firm i is then a subset $S_i \subset N$, with the convention that $i \in S_i$.
- (2) Given a profile of strategies $\mathbf{S} \equiv (S_i)_{i \in N}$, a connected component (can also think of it as a coalition) $\mathcal{C} \subset N$ is formed if, and only if, $S_i = \mathcal{C}$ for all $i \in \mathcal{C}$. For any component $\hat{\mathcal{C}}$ that is actually formed, the pattern of linkages among the firms in $\hat{\mathcal{C}}$ is implemented that allows to minimize the probability of default of any individual firm present in the component. Note that this pattern is the same as the one obtained as the solution of problem $[\mathbf{P}(K)]$, when the number of firms in $\hat{\mathcal{C}}$ is $K + 1$, and hence the probability of default of any firm in $\hat{\mathcal{C}}$ is just $V(K)$.

On the other hand if, for any firm i , S_i is such that $S_j \neq S_i$ for some $j \in S_i$, i 's proposal is taken to be rejected and hence firm i forms no linkages (i.e. remains in isolation).

It is clear that in the network formation game described in (1)-(2) above all the emphasis lies in the formation of the connected components in the system, that is in the pattern of segmentation of the system. On the other hand, the density of the connections within each component is chosen optimally, so as to minimize the probability of default of any firm in the component.

To study the network formation game given by (1)-(2) above, one possibility is to resort to the standard notion of Nash Equilibrium. However, an undesirable feature of such concept in this context is that it leads to a vast multiplicity of equilibrium networks. This is a consequence of the fact that the formation of any particular component requires the mutual agreement of all firms involved, which induces a coordination problem between the firms.³⁰ To address the entailed multiplicity issue, it is common in the network literature to consider a strengthening of the Nash equilibrium notion that reduces miscoordination by allowing sets of firms to deviate jointly (see e.g. Goyal and Vega Redondo (2007) or Calvó-Armengol and Ikiliç (2009)). In the context of our model we shall introduce this possibility by means of the concept we label *Coalition-Proof*

³⁰As an extreme illustration, note that the empty network, where each firm stays in isolation, can always be supported by a Nash equilibrium in which no firm makes a proposal to link to others.

Equilibrium (CPE) where any group (i.e. coalition) of firms can coordinate their deviations. To state it formally, for any given strategy profile $\mathbf{S} = (S_i)_{i \in N}$ we let $\mathcal{C}_j(\mathbf{S})$ denote the component (induced by \mathbf{S}) to which firm j belongs and, with a slight abuse of the previous notation, we write $V_j(\mathbf{S}) \equiv V(|\mathcal{C}_j(\mathbf{S})|)$ to denote the probability of default of firm j in this situation, when indirectly hit by a b shock.

DEFINITION 1 *A strategy profile $\mathbf{S} \equiv (S_i)_{i \in N}$ of the network-formation game described in (1)-(2) defines a Coalition-Proof Equilibrium (CPE) if there is no subset of firms $W \subseteq N$ and a strategy profile for these firms, $(S'_j)_{j \in W}$, such that*

$$\forall i \in W, \quad V_i \left((S'_j)_{j \in W}, (S_k)_{k \in N \setminus W} \right) < V_i(\mathbf{S}).$$

As motivated by our previous discussion, the *CPE* concept allows not just single firms but also arbitrary collections of them to implement jointly profitable deviations (that allow to reduce the probability of defaults of the firms deviating). This remedies the coordination problems that, as explained, render Nash equilibria an ineffective theoretical concept in this case. In essence, CPE can be seen as a refinement of Nash Equilibrium that addresses the latter's vast multiplicity by postulating that it must be immune to a wider set of deviations.

As it turns out, the relationship between equilibria and social optima is quite straightforward for the scenarios considered in Propositions 1 and 4. Recall that, for these results, the shock distribution was respectively taken to be as follows: (i) a Pareto distribution (or, more generally, one that induces a default-probability function $F(\cdot)$ with uniform curvature); (ii) a suitable mixture of a Pareto distribution with thin tails (or one that induces a *convex* default-probability function) and a Dirac measure. For each of these two scenarios, the following result establishes an essential equivalence between equilibrium and optimal configurations.

PROPOSITION 5 *Assume the conditions on the shock distribution specified in either Proposition 1 or 4. Let $A^* \equiv [(a_{ij}^*)_{j \neq i}]_{i=1}^N$ be a matrix of exposures that is socially optimal (i.e. solving problem [SP]). Then, there exists a CPE $\hat{\mathbf{S}} \equiv (\hat{S}_i)_{i \in N}$ of the network-formation game that yields an exposure matrix \hat{A} equal to A^* . Conversely, for any exposure matrix $\hat{A} \equiv [(\hat{a}_{ij})_{j \neq i}]_{i=1}^N$ obtained at a CPE, \hat{A} is also socially optimal.*

The reason why the notions of equilibrium and social optimality yield the same outcomes in the scenarios considered above is that, in those cases, the optimization problems [SP] and [IP] have the same solution. As we saw in the proofs of these propositions, the additional constraints present in problem [SP], (9) and (10), are in fact not binding. Given the specification of problem [IP] it is clear that no subset W of firms can attain a level of the individual probability of default lower than the one attained at a solution of [IP].

Hence the matrix of exposures obtained at a solution of [SP] (and [IP]) must also be a CPE outcome, since there can be no profitable deviation. Similarly, the only possible CPE outcome is given by the solution of [SP], as otherwise a profitable deviation would exist: for instance, when the shock distribution is as in the situation considered in Proposition 4, if firms were divided into two or more disjoint components, the set $W = N$ formed by all the firms in the system has a profitable deviation.

The above argument suggests that, in our model, any tension between social optimality and individual incentives may only arise because of the additional constraints in problem [SP], that relate among them the intensity of the linkages of the various firms so as to ensure the feasibility of the overall financial structure. Whenever these constraints bind, the pattern of linkages must be adjusted so that feasibility holds. Proposition 3. considers indeed a scenario where this situation arises and characterizes the socially optimal way to proceed in that case. We show in the following result that the individual response of firms, at a CPE, is markedly different:

PROPOSITION 6 *Under the same assumptions of Proposition 3, the equilibrium matrix of exposures \hat{A} differs from the socially optimal matrix A^* : \hat{A} exhibits one component of size $\hat{K} + 1$, with exposures equal to \hat{a} , and a second component, of lower size $N - \hat{K} - 1$, with exposures equal to $a' = (1 - \alpha) / (N - \bar{K} - 1) > \hat{a}$.*

Proof. See the Appendix.

We show in the proof that when we have two components which are both of size smaller than $\hat{K} + 1$, as at socially optimum structure A^* , a set of $\hat{K} + 1$ firms has a profitable deviation. Hence at a CPE one component in the system must be at the individually optimal size (that is, solving problem [IP]), with \hat{K} nonzero off diagonal terms all equal to \hat{a} . Firms in this component achieve the minimal default probability, and for this reason they are not willing to deviate, since they would do worse in any other possible configuration. The remaining firms lie in a smaller component, hence their default probability is higher, but they also cannot find a profitable deviation. Even though they would benefit by forming a larger component, they are unable to find other firms willing to join them since all other firms are already at the individually optimal size.³¹ Hence, at a CPE the feasibility constraints are met by adjusting the pattern of linkages of a single component, which must absorb then all the slack and feature a higher default probability. In contrast, at a social optimum, as we saw, the linkages of all firms in all components are modified, preserving the symmetry among the components in the system.

³¹Note that the deviations contemplated in the proof of Proposition 6 are “internally consistent” in the following sense. Given any set of firms that find it optimal to deviate, this deviation is itself in the interest of any subset of this set that might reconsider the situation. This requirement (commonly demanded in the game-theoretic literature for coalition-based notions of equilibrium) is clearly satisfied in our case since any firm in a subset of firms who refuses to follow suit with a deviation by $\hat{K} + 1$ (or fewer) firms can only lose, since the function $V(\cdot)$ is monotonically decreasing for $K \leq \hat{K}$.

This discrepancy between equilibrium and social optimum implies that the expected number of defaults at a CPE is now higher than at the social optimum. The source of the inefficiency lies in the fact that some of the firms form a component that is too large from the viewpoint of social optimality. Certainly, forming such a component is individually optimal for all the firms involved. However, in doing so they do not take into account the following externality: given the constraint imposed by the total number of firms in the system, this component can only be formed by forcing some other component to remain too small. The contrast between social and individual optimality is, as usual, a reflection of a non-internalized externality – in this case, the fact that the feasibility of the financial structure imposes a relationship among the linkages of all individual firms, in all components. As a simple illustration of the problem, we present the following example.

EXAMPLE 1 *Set $\gamma_1 = 2$, $\gamma_2 = 0.5$, $p = 0.9541$, $\alpha = 0.5$, $N = 10$, $\underline{L} = R - M$. For these values, we find that the individually optimal component size, which solves $[\mathbf{IP}]$, is $\hat{K} + 1 = 7$ and the pattern of exposures within the component is uniform, equal to $\hat{a} = \frac{0.5}{6}$. The corresponding fraction of firms defaulting in such a component, when indirectly hit by a b shock (conditionally on such a shock hitting the system) is then 0.119.*

When $N = 10$ it is clearly not feasible to have all components at the optimal size of 7. In accord with Proposition 3, at the social optimum we have two components both of size 5 and uniform exposure within each of them. The expected number of defaults³² at the social optimum is then 1.245. In contrast, at a CPE the population is arranged into a component of size 7 and another one of size 3, where the fraction of firms defaulting is higher and equal to 0.165. The expected number of defaults at the CPE is then $7 \cdot 0.119 + 3 \cdot 0.165 = 1.328$, higher than at the social optimum, showing the inefficiency of the equilibrium.

The existence of a conflict between efficiency and equilibrium is of course hardly novel nor surprising in the field of social networks (see e.g. Jackson and Wolinsky (1996) for an early exploration of this conflict). For, typically, the creation or destruction of any link between two agents impose externalities on others that are not internalized by the two agents involved in the linking decision. In the context of risk-sharing, this tension has been studied in a recent paper by Bramoullé and Kranton (2007) – hereafter labeled BK – and it is interesting to understand its difference with our approach. We close, therefore, this section with a brief comparison of the two models.

BK consider an environment consisting of a finite number of agents affected by i.i.d. income shocks. Linkages generate risk sharing in a way similar to our model, except in two important respects: (a) risk-

³²More precisely, this is again the expected number of firms defaulting when indirectly hit by a b shock, conditionally on such a shock hitting a firm in the system.

sharing is complete (i.e. uniform) across all members in a component; (b) there is no costly default, so the size of optimal components is just limited by the fact that links are assumed to be costly. Focusing on the notion of strategic stability (which is weaker than ours),³³ BK are also interested in comparing efficient and equilibrium configurations. They find that whenever equilibrium structures exist (not always), there are at most two asymmetric components, with sizes smaller than the optimal one.

The contrast between BK’s conclusions and ours derives from the nature of the externality in the two cases. In BK, given that the cost of any new link is borne only by the two agents involved, the equilibrium induces an underinvestment in link formation (which is a “public good”). Instead, in our case there are no linking costs, so the nature of the externality that is not internalized is quite different. It has to do with the fact that, when firms deviate to join components of larger size than the socially optimal one, they do not internalize that the firms that are left behind will be forced to form components that are inefficiently small. Thus, in the end, it is the need to meet an overall feasibility constraint imposed by the finite measure of the population which typically generates inefficiencies.

5 Asymmetric environments

So far we have concentrated the discussion on a situation where all firms in the system are *ex ante* identical. Although this allows us to obtain clear-cut analytic results and gain some understanding of the forces at play, it is important to extend our analysis to situations where, as it often happens in the real world, firms are significantly different in their size or in the shock distribution they face. This is the purpose of this section, which we divide into two parts. First, Subsection 5.1 considers the case where the asymmetry between firms pertains to the shock distribution; then, in Subsection 5.2, we turn to the case where the asymmetry concerns the firms’ size. We find that the conclusion for the two types of asymmetries is somewhat different. In the first case, whenever some segmentation is beneficial, the optimal financial structure exhibits firms divided into homogeneous components. It is then never optimal to have firms with different risk profiles being exposed to each other. On the other hand, when firms only differ in size, that is in the level of their resources and the likelihood and/or magnitude of the shocks hitting them, an irrelevance result holds: once the pattern of firms’ exposures is suitably rescaled, heterogeneous components feature the same default probability as homogeneous ones.

³³Strategic stability allows only for coalitions of at most two players and rules out as well the simultaneous creation and destruction of links. See Jackson and Wolinsky (1996) for details.

5.1 Shock asymmetry

We allow now the distribution to differ across firms. More precisely, we assume the N firms are partitioned into T subsets N_1, \dots, N_T , so that $N = \cup_{t=1}^T N_t$, and for every firm $i \in N_t$ the b shock follows a cumulative distribution function Φ_t , for all $t \in \{1, \dots, T\}$. The distributions Φ_t have the same expected value but may otherwise differ in their risk characteristics. We maintain then the assumptions that the return on all firms' projects is equal to R in the absence of shocks and to $R - \underline{L}$ if a small shock hits and that the b shocks hit any firm with probability $q(1 - \pi)$.

Consider first the value of the expected fraction of firms who default, if indirectly hit by a shock b , in a homogenous component c with $K^c + 1$ firms all of the same type t , when the intensity of linkages in the component is optimally chosen. This is simply given by the value of the objective function at a solution of problem $[\mathbf{P}(K)]$ with $K = K^c$ and $F(\cdot) = 1 - \Phi_t((R - M)/\cdot)$; we denote then this value by $V(K^c, \Phi_t)$ for $t = 1, \dots, T$. On this basis we can find a *lower bound* for the corresponding expression $V(K^c, (|N_t^c|, \Phi_t)_{t=1}^T)$ for a heterogeneous component, of the same size $K^c + 1$, but now with a subset N_t^c of firms of type $t = 1, \dots, T$:

$$V(K^c, (|N_t^c|, \Phi_t)_{t=1}^T) \geq \sum_{t=1}^T \frac{|N_t^c|}{K^c + 1} V(K^c, \Phi_t), \quad (18)$$

where $|\cdot|$ stands for the cardinality of the set in question. To understand the expression on the right hand side of (18), note that $|N_t^c| / (K^c + 1)$ is the probability that, if a b shock hits the component, it hits a firm of type t . When this happens $V(K^c, \Phi_t)$ is the expected fraction of firms indirectly hit who default, if the intensity of the linkages is optimally chosen (as in problem $[\mathbf{P}(K)]$) for a shock with distribution Φ_t . This observation uses the fact that, since firms of all types have the same size, they have the same capacity for absorbing shocks and this implies that the expected fraction of firms defaulting only depends on the size of the component and the type t of the firm hit by a b shock. Hence the specific type of a firm that is only indirectly hit by a shock does not matter for its default probability. It is important to observe that the value on the right hand side of (18) is obtained by choosing optimally the intensity of linkages for each t . On the other hand, since the intensity of linkages can only take one value which cannot be adjusted according to the type t of firm hit by a b shocks, this expression only constitutes a lower bound for $V(K^c, (|N_t^c|, \Phi_t)_{t=1}^T)$.

We are interested in particular in identifying the optimal segmentation structure, which now requires determining not only the number of firms that should lie in each component but also - most importantly - the composition of each component in terms of the different types of firms included in it. One important concern will be to understand whether firms' matching within components should be assortative or disassortative -

that is, whether components should be more or less homogeneous. In this context, therefore, a structure must specify a set of components $\{1, \dots, C\}$ and, for each component c , the corresponding type distribution $\{N_t^c\}_{t=1, \dots, T}$, where $N_t^c \subseteq N_t$ stands for the subset of firms of type t that belong to component c . Let \hat{K}_t be the number of nonzero terms obtained at a solution of problem [IP] when the distribution of the b shocks is Φ_t for all firms; as we saw³⁴ in Section 3, \hat{K}_t is the value of K at which the function $V(K, \Phi_t)$ reaches a minimal value. Thus, we have:

PROPOSITION 7 *Suppose the same assumptions of Proposition 2 hold. Suppose also that, for every t , the cardinality $|N_t|$ of the set of firms of type t is a multiple of $\hat{K}_t + 1$. Then the optimal financial structure features firms belonging to homogeneous components, in each of which firms are all of the same type, and $\hat{K}_t + 1$ is the common size of every component consisting of firms of any given type t .*

Proof. See the Appendix.

The intuition for this result is as follows. Conditionally on the b shock hitting a firm of type t , the optimal arrangement is to have all components of size $\hat{K}_t + 1$, since \hat{K}_t is the value that minimizes $V(K, \Phi_t)$, and the pattern of linkages set at the values obtained at a solution of problem [IP] with $F(.) = 1 - \Phi_t((R - M)/.)$. If we had an heterogeneous component, with firms of type t' , in addition to t , present in it, the size of this component - and the pattern of its connections - could not be set to equal *simultaneously* the (generally different) optimal values for the cases of shocks hitting type t and type t' firms. Hence the expected number of defaults could be reduced by rearranging firms within homogeneous components, as this would allow to attain the minimum of $V(K, \Phi_t)$ for each t (and, under the stated assumptions, this is always feasible).

The result says that, when the risk profiles of the shocks affecting different firms are sufficiently diverse so that the optimal degree of segmentation is not the same across profiles, the optimal structure exhibits perfectly assortative matching, that is only components of homogeneous firms are formed. Of course, a key feature required for this conclusion to hold is that, as already noticed, the expected number of indirect defaults in a component only depends on the type t of the firm directly hit by the shock and the size of the component. That is, the indirect effect of the shock does *not* depend on the distribution of types among the firms in the component not directly hit by the shock.

³⁴See in particular the proof of Proposition 3 in the Appendix.

5.2 Size asymmetry

Now we turn our attention to the case where firms are heterogeneous in size. For simplicity, let us consider a situation with just two possible sizes. On the one hand, there are firms of unit size, identical to the ones we have been considering so far. On the other hand, there are firms of size $\beta > 1$, with such larger size having the following two implications. First, the return on the projects of these larger firms when no shock hits them is βR , i.e. it is scaled up by β compared to that of the smaller firms (naturally the same factor β applies to the value of these firms' liabilities, which are then βM). Second, the larger firms face a probability of being directly hit by a shock that is β times larger (that is, equal to βq). Thus the present scenario, where size affects the probability of arrival of the shocks, *not* their distribution, can be viewed as complementary to the one considered in the previous Subsection 5.1. Also, here one can view a large firm as a merger of β unit-sized firms.

Of course, condition (4), ensuring some balance between the exposure of a firm to other firms, and from the other firms to this firm, needs to be properly reformulated when we consider exposures between firms of different size. The level of the exposure of a firm of size 1 to a large firm (of size β) should 'count' as β times an exposure of the same level to a firm of unit size (because of the higher probability of a shock). Vice versa, for a large firm the exposure to a smaller (unit size) firm counts as $1/\beta$ times the exposure to a firm of the same size. Let us denote by N_1 the subset of firms of size 1, and by N_β the subset of firms of size β .

Similarly, N_1^c and N_β^c stand for the corresponding subsets of small and big firms within a component c . This leads to an *effective* size of component c given by $K^c + 1 = |N_1^c| + \beta |N_\beta^c|$. In a component of this kind the expected fraction of firms who default (again in terms of their effective size), if indirectly hit by a shock b , is given by

$$\begin{aligned} & \frac{1}{K^c + 1} \left[\sum_{i \in N_1^c} \left(\sum_{j \in N_1^c \setminus i} F(a_{ij}) + \beta \sum_{j \in N_\beta^c} \Pr \left\{ L > \frac{(R - M)}{a_{ij}} \right\} \right) + \right. \\ & \left. \sum_{i \in N_\beta^c} \beta \left(\sum_{j \in N_1^c} \Pr \left\{ L > \frac{\beta(R - M)}{a_{ij}} \right\} + \beta \sum_{j \in N_\beta^c / i} \Pr \left\{ L > \frac{\beta(R - M)}{a_{ij}} \right\} \right) \right] \end{aligned} \quad (19)$$

Let $V \left(K^c, |N_1^c|, |N_\beta^c| \right)$ denote the value of the above expression when the pattern of linkages is chosen optimally, subject to constraints (9) (suitably adjusted to reflect the different effective sizes of the two types of firms) and (10). We show in the next proposition that in this case mixed components fare equally well, in terms of expected defaults, as homogeneous components. We have in fact:

PROPOSITION 8 *Under any of the conditions on the distribution of the b shocks stated in Propositions 1, 2, or 4, for all $1 \leq K^c \leq N - 1$ and all subsets $N_1^c \subseteq N_1$ and $N_\beta^c \subseteq N_\beta$ such that $|N_1^c| + \beta |N_\beta^c| = K^c + 1$, we have $V(K^c, |N_1^c|, |N_\beta^c|) = V(K^c)$.*

The proposition establishes an irrelevance result: the expected fraction of defaults in a component containing both large and unit sized firms is exactly the same as that in a component (of the same effective size) where all firms have unit size, provided in each of them the pattern of exposures is chosen optimally, subject to the constraints. The proof shows that, for any pattern of exposures $[(\bar{a}_{ij})_{j \neq i}]_{i=1}^{K^c+1}$ in a homogeneous component of smaller firms, the same level of expected defaults can be attained in a mixed component, by appropriately rescaling the exposures of larger firms, setting $a_{ij} = \beta \bar{a}_{ij}$ for all $i \in N_\beta^c$ and all j , while keeping unchanged the levels of exposures of smaller firms, $a_{kj} = \bar{a}_{kj}$ for all $k \in N_1^c$ and all j .

The result can be understood in the light of the fact that we have constructed larger firms to be effectively the merger of β smaller firms. It is interesting to note that an analogous result holds for other possible specifications of the effects of firms' size (for instance if β affects the size of the shock, which is then given by βl , instead of the probability of being hit by a shock, or even a combination of the two). In this case the shock hitting a large firm is no longer the same as the one hitting β small firms, but the equivalence result still holds, only the rescaling factors change.

We can say that Proposition 8 holds for reasons that are reminiscent of those behind the Modigliani-Miller theorem. The result shows that, as far as the effects on risk sharing and contagion are concerned, smaller firms are indifferent between forming linkages with firms of the same size or with larger firms. This stands clearly in contrast to the case where firms differ in terms of their risk characteristics, as we saw in the previous section, and has some important implications. In particular, the presence of any other, even small, advantage for small firms to connect to larger firms, based on other considerations than risk sharing, for instance informational processing reasons (say, if information acquisition has increasing returns to scale), will lead to the formation of components where large and small firms coexist, as in the case of core periphery networks.

6 Conclusion

We have proposed a stylized model to study the problem that arises when firms establish linkages among them to weather shocks that can threaten their survival, but by so doing they become exposed to the risk coming from those same connections. We have then characterized the patterns of linkages among firms that

allow to minimize default in the system for different properties of the distribution of the shocks that may hit firms. For distributions that belong to a canonical class including Pareto distributions we showed that a uniform level of exposure to other firms is always beneficial and the optimal response to an increased likelihood of large shocks is the segmentation of the system into disjoint components. In contrast, for distributions with mass points the optimal response is to have different levels of exposure to other firms, i.e. sparser connectivity.

We have also explored whether such social optimum is aligned with the choices of individual firms to establish linkages with other firms. There is a conflict between individual incentives and social optima, due to the fact that firms have always an incentive to form connected components of the size that minimizes the default probability of their members, thus ignoring the negative externality this behavior sometimes imposes on other firms. Finally, we have found that when we allow for heterogeneity among firms in the distribution of shocks they receive, optimality requires perfect assortativity, that is to have homogeneous components, in each of which the risk characteristics of the shocks affecting firms is the same. In contrast, with heterogeneity in firms' size an irrelevance result holds, implying the optimality of components where large and small firms coexist.

The basic insights obtained from this analysis can inform the regulation and design of financial systems in the real world. Should one separate commercial and investment banking? Should the financial systems of different regions be insulated from each other, or should overall integration be pursued? And, in the end, how effectively can we trust any prescription along these lines to be indeed implemented by banks (i.e. to be compatible with their own individual incentives)? These are some of the questions that naturally and crucially arise, and to which we have shown our model can provide answers based on the analysis of some key factors at play.

As mentioned in the Introduction, Loepfe et al. (2013) extends our analysis through numerical methods to environments with more realistic features and verifies the robustness of our main results. More precisely, it finds that the optimality of forming smaller/less dense networks when shocks come from distributions that put more weight on large shocks extend to the cases where we allow for the possibility of multiple shocks hitting simultaneously firms, for other kinds of shock distributions and for networks where the balance condition (4) may not hold. In a different vein, they also consider the effects of shocks to the real-life network of corporate control studied by Battiston et al. (2012) and find that they depend in important ways on the nature of the shock distribution as suggested by our theory.

But, of course, a proper discussion of the risk-sharing and contagion phenomena in the real world must

also account for many important dimensions that our model abstracts from. One of them is the consideration of problems of moral hazard and, in general, asymmetries of information that have been repeatedly singled out as a key factor underlying the recent financial crisis. We also ignore forces – intermediation, for instance, or also cooperation and the exploitation of synergies – that, beyond risk-sharing and limiting contagion, contribute to determining the form of financial networks. A proper study of all these crucial issues will demand a richer theoretical framework and a more powerful methodology, the development of which can hopefully build in a fruitful manner upon the present effort.

Appendix

Proof of Proposition 1

Let us consider problem **[IP]**. Note that this problem always has a solution. Assume first that $F(\cdot)$ is convex and suppose that, at a solution of **[IP]** we have $a_{ij} \neq a_{ik}$ for $j, k \neq i$. By the convexity of $F(\cdot)$ we have:

$$2F\left(\frac{a_{ij} + a_{ik}}{2}\right) < F(a_{ij}) + F(a_{ik}).$$

Hence replacing both a_{ij} and a_{ik} with $(a_{ij} + a_{ik})/2$ still satisfies the constraints of problem **[IP]** and yields a strictly lower value of the objective function, thus a contradiction. It then follows that at a solution of **[IP]** we must have $a_{ij} = a_{ik}$ for all $j, k \neq i$ and hence $a_{ij} = \frac{1}{2(N-1)} \quad \forall j \neq i$. Since it is symmetric and all entries are strictly positive this solution also satisfies the additional constraints present in problem **[SP]** and so is also a solution of **[SP]**. It then follows that at a social optimum we have a single component of size N with uniform exposure.

Assume next that $F(\cdot)$ is concave and suppose that, at a solution of problem **[IP]** $a_{ij} \neq 0$, $a_{ik} \neq 0$ for some $j, k \neq i$. By the concavity of $F(\cdot)$ we get:

$$\begin{aligned} \frac{a_{ij}}{a_{ij} + a_{ik}} F(a_{ij} + a_{ik}) + \frac{a_{ik}}{a_{ij} + a_{ik}} F(0) &< F(a_{ij}), \\ \frac{a_{ik}}{a_{ij} + a_{ik}} F(a_{ij} + a_{ik}) + \frac{a_{ij}}{a_{ij} + a_{ik}} F(0) &< F(a_{ik}). \end{aligned}$$

Summing the two inequalities yields:

$$F(a_{ij} + a_{ik}) + F(0) < F(a_{ij}) + F(a_{ik}),$$

hence replacing a_{ij}, a_{ik} with the pair $(a_{ij} + a_{ik}), 0$ is still a feasible configuration for problem [IP] and allows to lower the value of the objective function, again a contradiction. Thus at a solution of [IP] we can only have one nonzero off diagonal entry of A , that is $a_{ij} = 1 - \alpha$ for some $j \neq i$. Since N is even, we can always select the element of each row that is equal to $1 - \alpha$ so as to satisfy the balance condition (4) and the condition that, within the component, every firm is linked to every other firm lying in it: set A symmetric, with diagonal blocks of dimension 2. Hence this specification also solves problem [SP]. \square

Proof of Proposition 2

Under the stated assumption on $\phi(l)$, the function $F(a_{ij})$ takes the following form: $pa_{ij}^{\gamma_1} \left(\frac{L}{R-M}\right)^{\gamma_1} + (1-p)a_{ij}^{\gamma_2} \left(\frac{L}{R-M}\right)^{\gamma_2}$. Differentiating this function with respect to a_{ij} we obtain the following expression for the first order conditions for a solution of problem [IP] for all j for which $a_{ij} > 0$:

$$F'(a_{ij}) = p\gamma_1 a_{1j}^{\gamma_1-1} \left(\frac{L}{R-M}\right)^{\gamma_1} + (1-p)\gamma_2 a_{1j}^{\gamma_2-1} \left(\frac{L}{R-M}\right)^{\gamma_2} = \lambda \quad (20)$$

where λ is the Lagrange multiplier of constraint (11). Since $\gamma_1 > 1$ and $\gamma_2 < 1$ the inequality

$$F''(a_{ij}) = p\gamma_1(\gamma_1 - 1)a_{1j}^{\gamma_1-2} \left(\frac{L}{R-M}\right)^{\gamma_1} + (1-p)\gamma_2(\gamma_2 - 1)a_{1j}^{\gamma_2-2} \left(\frac{L}{R-M}\right)^{\gamma_2} > 0$$

holds if and only if

$$a_{1j} > a^{**} \equiv \left(\frac{(1-p)\gamma_2(1-\gamma_2)}{p\gamma_1(\gamma_1-1)}\right)^{\frac{1}{\gamma_1-\gamma_2}} \left(\frac{R-M}{L}\right). \quad (21)$$

Thus the function $F'(a_{ij})$ is first decreasing and then increasing.

Let $(\hat{a}_{ij})_{j \neq i}$ be a solution of problem [IP]. Since we have seen that for all $j \neq i$ such that $\hat{a}_{ij} > 0$ we must have $F'(\hat{a}_{ij}) = \lambda$ and $F'(a_{ij})$ is first decreasing and then increasing, there are at most two values $a_2 > a_1 > 0$ for which the first order condition (20) is satisfied, that is $\hat{a}_{ij} \in \{a_1, a_2\}$. Moreover, $F'''(a_1) < 0$ whereas $F''(a_2) > 0$ so that, for any small $\Delta > 0$:

$$F(a_1 + \Delta) + F(a_1 - \Delta) - 2F(a_1) \simeq F'(a_1)(\Delta - \Delta) + F''(a_1)\Delta^2 < 0.$$

Hence we cannot have two distinct values $j, k \neq i$ such that $\hat{a}_{ij} = \hat{a}_{ik} = a_1$ (if so, a lower value of the objective function could be achieved by increasing a_{ij} and decreasing a_{ik} by the same amount Δ and this is always a feasible change).

We have thus established that that there are at most two positive solutions a_1, a_2 , the first one repeated

- at most - once, the second one repeated a number $K \geq 1$ of times, satisfying (21). Hence the number of active linkages is K or $K + 1$. Using this fact, constraint (11) can be rewritten as

$$Ka_2 + a_1 = 1 - \alpha,$$

In the situation under consideration we can then simplify the individual problem **[IP]** as follows:

$$\begin{aligned} \min_{K, a_1, a_2} \quad & KF(a_2) + F(a_1) \\ \text{s.t.} \quad & Ka_2 + a_1 = 1 - \alpha \\ & a_2 \geq a_1 \geq 0. \end{aligned} \tag{22}$$

Note that in (22) we have ignored the constraint that K should lie between³⁵ 0 and $N - 1$ and we will then have to verify such constraint is satisfied to claim that a solution of (22) also solves **[IP]**.

There are three possible types of solutions of problem (22):

1. $a_1 = 0$, so there are K active linkages for each firm, all with the same intensity;
2. $a_1 = a_2$, in which case the number of active linkages is $K + 1$ and again there is uniform exposure to all other firms to which a firm is connected;
3. $a_2 > a_1 > 0$, hence there are $K + 1$ active linkages, but exposure is no longer uniform (K linkages have intensity a_2 and one has lower intensity a_1).

In the corner solution described in case 1. the FOC's of problem (22) are as follows

$$\begin{aligned} KF'(a_2) &= \lambda K \\ F'(0) &> \lambda \\ F(a_2) &= \lambda a_2 \\ Ka_2 &= 1 - \alpha \end{aligned}$$

³⁵As stated in the main text, we ignore for simplicity the integer constraint regarding the value of K .

Using the fact that $F'(0) = \infty > \lambda$ is always satisfied, the above system reduces to

$$\begin{aligned} F'(a_2) &= \lambda \\ F(a_2) &= \lambda a_2 \\ K a_2 &= 1 - \alpha \end{aligned} \tag{23}$$

Hence we get:

$$F' \left(\frac{1 - \alpha}{K} \right) = \frac{F \left(\frac{1 - \alpha}{K} \right)}{\frac{1 - \alpha}{K}}.$$

In case 2. the FOC's reduce to

$$\begin{aligned} F'(a_2) &= \lambda \\ F(a_2) &= \lambda a_2 \\ (K + 1)a_2 &= 1 - \alpha \end{aligned} \tag{24}$$

that is,

$$F' \left(\frac{1 - \alpha}{K + 1} \right) = \frac{F \left(\frac{1 - \alpha}{K + 1} \right)}{\frac{1 - \alpha}{K + 1}},$$

analogous to (23).

Finally, in the interior solution described in case 3. the FOC's can be written as follows:

$$\begin{aligned} F'(a_2) &= F'(a_1) \\ \frac{F(a_2)}{a_2} &= F'(a_2) \\ K a_2 + a_1 &= 1 - \alpha \end{aligned} \tag{25}$$

or

$$\begin{aligned} \frac{F(a_2)}{a_2} &= F'(a_2) = F'(a_1) \\ K a_2 + a_1 &= 1 - \alpha \end{aligned}$$

Let K^{int} be the value of K satisfying the FOC's in (25) for the interior solution of case 3. and K^c the solution of the FOC's for the corner solution of case 1. in (23) (as argued above, $K^c - 1$ then also solves

the FOC's for case 2.). We now show that only the corner solution of case 1. (or 2.) can be optimal. We establish this by contradiction.

Since both in the corner and in the interior solution we have $a_2 = F(a_2)/F'(a_2)$, a_2 has the same value in these two solutions³⁶, given by:

$$a_2 = \hat{a} = \left(\frac{(1-p)(1-\gamma_2)}{p(\gamma_1-1)} \right)^{\frac{1}{\gamma_1-\gamma_2}} \left(\frac{R-M}{\underline{L}} \right). \quad (26)$$

Note that $\hat{a} > a^{**}$, hence $F''(\hat{a}) > 0$. The interior solution solves then problem (22) if

$$K^{int} F(a_2) + F(a_1) \leq K^c F(a_2),$$

or

$$F(a_1) \leq (K^c - K^{int}) F(a_2) \quad (27)$$

From the last equation in the FOC's of case 3., (25), using $a_2 = F(a_2)/F'(a_2)$, we get

$$K^{int} \frac{F(a_2)}{F'(a_2)} + a_1 = 1 - \alpha. \quad (28)$$

From the corresponding equation in (23) we also get $a_2 = (1 - \alpha)/K^c$ and so the above equation (28) can be rewritten as:

$$(K^c - K^{int}) F(a_2) = F'(a_2) a_1. \quad (29)$$

Combining (29) with (27), and using the fact that from (25) we also get $F'(a_2) = F'(a_1)$, yields

$$F'(a_1) a_1 \geq F(a_1)$$

³⁶Substituting the expression of $F(\cdot)$ in the equation $a_2 = F(a_2)/F'(a_2)$ we obtain in fact:

$$p\gamma_1 a_2^{\gamma_1-1} \left(\frac{\underline{L}}{R-M} \right)^{\gamma_1} + (1-p)\gamma_2 a_2^{\gamma_2-1} \left(\frac{\underline{L}}{R-M} \right)^{\gamma_2} = \frac{p a_2^{\gamma_1} \left(\frac{\underline{L}}{R-M} \right)^{\gamma_1} + (1-p) a_2^{\gamma_2} \left(\frac{\underline{L}}{R-M} \right)^{\gamma_2}}{a_2}$$

which has a unique solution for a_2 given by the expression in (26).

or

$$\begin{aligned}
p\gamma_1 a_1^{\gamma_1} \left(\frac{\underline{L}}{R-M} \right)^{\gamma_1} + (1-p)\gamma_2 a_1^{\gamma_2} \left(\frac{\underline{L}}{R-M} \right)^{\gamma_2} &\geq p a_1^{\gamma_1} \left(\frac{\underline{L}}{R-M} \right)^{\gamma_1} + (1-p) a_1^{\gamma_2} \left(\frac{\underline{L}}{R-M} \right)^{\gamma_2} \\
&\Leftrightarrow (1-p)(1-\gamma_2) a_1^{\gamma_2} \left(\frac{\underline{L}}{R-M} \right)^{\gamma_2} \leq p a_1^{\gamma_1} (\gamma_1 - 1) \left(\frac{\underline{L}}{R-M} \right)^{\gamma_1} \\
&\Leftrightarrow \left[\frac{(1-p)(1-\gamma_2)}{p(\gamma_1 - 1)} \right]^{\frac{1}{\gamma_1 - \gamma_2}} \left(\frac{R-M}{\underline{L}} \right) \leq a_1.
\end{aligned}$$

From (26) we see that the term on the left hand side of the last inequality is equal to $\hat{a} = a_2$, hence we have $a_2 \leq a_1$, a contradiction. There is so no interior solution of problem (22), the solution obtains at a corner, and is characterized by (23): $a_2 = \hat{a}$ and $\hat{K} = K^c = (1 - \alpha) / \hat{a}$. Since, under the condition stated in the proposition, $K^c \in (1, N - 1)$, this solution also solves problem [IP]. \square

Proof of Proposition 3

To characterize the solution of problem [SP] in this case, it is convenient to consider the following constrained version of problem [IP], where the number of off-diagonal elements is constrained to be \bar{K} , with $1 \leq \bar{K} < N - 1$:

$$\begin{aligned}
[\mathbf{IP}(\bar{K})] \quad & \min_{(a_{ij})_{j \neq i}} \sum_{j \neq i}^{\bar{K}+1} F(a_{ij}) \quad \text{s. t.} \\
& \sum_{j \neq i}^{\bar{K}+1} a_{ij} = 1 - \alpha \\
& a_{ij} \geq 0, \quad \forall j \neq i
\end{aligned}$$

Let us denote by $D(\bar{K})$ the value of $\sum_{j \neq i} F(a_{ij})$ at a configuration that solves problem $[\mathbf{IP}(\bar{K})]$, for some given \bar{K} .

Applying again the same method followed in analyzing problem [IP] in the proof of Proposition 2, the solution of $[\mathbf{IP}(\bar{K})]$ is obtained by considering the following simplified constrained problem:

$$\begin{aligned}
D(\bar{K}) &\equiv \min_{a_2} (K - 1) F(a_2) + F(a_1) \\
\text{s.t. } (K - 1) a_2 + a_1 &= 1 - \alpha \\
K &\leq \bar{K} \\
a_2 &\geq a_1 \geq 0
\end{aligned} \tag{30}$$

Evidently, for all $\bar{K} \geq K^c$ the constraint $K \leq \bar{K}$ is not binding and the optimum is the same as the one characterized in the previous proof, that is $D(\bar{K}) = K^c F((1 - \alpha)/K^c)$.

We show next that for $\bar{K} < K^c$ the solution of (30) obtains when $a_1 = a_2$, that is when there is uniform exposure to all other \bar{K} firms: $\bar{a}_2 = (1 - \alpha) / \bar{K}$. To see this, note first that the constraint $K \leq \bar{K}$ must be binding at a solution of (30). Using then the first constraint of problem (30) to substitute for a_1 in the objective function of the problem and differentiating with respect to a_2 , we get the following FOC:

$$(\bar{K} - 1) (F'(a_2) - F'(1 - \alpha - (\bar{K} - 1) a_2)) = 0. \quad (31)$$

This equation has always a solution for $\bar{a}_2 = (1 - \alpha) / \bar{K}$. Also, differentiating again the expression on the left hand side and evaluating the result at \bar{a}_2 yields

$$F''(\bar{a}_2) (\bar{K} - 1) \bar{K} > 0, \quad (32)$$

where the sign follows from the fact that when $\bar{K} < K^c$, $\bar{a}_2 > \hat{a}$ and we showed in the previous proof that \hat{a} satisfies (21) and so $F(\cdot)$ is convex for $a_2 > \hat{a}$.

As shown in Lemma 6 below, when $\gamma_1 \in [2, 3]$ there is only one other solution of (31), $a'_2 > (1 - \alpha) / \bar{K}$. When there are only two solutions of the FOC given by (31), since the function we are minimizing is convex at the solution given by \bar{a}_2 , as shown in (32), it follows that the other critical point a'_2 cannot be a local minimum.

Finally, note that \bar{a}_2 is the lowest value of a_2 satisfying the constraints of the above problem (30). The highest value is $a_2 = (1 - \alpha) / (\bar{K} - 1)$, so that $a_1 = 0$, and this also is not a minimum since $F((1 - \alpha) / (\bar{K} - 1)) (\bar{K} - 1) > F((1 - \alpha) / \bar{K}) \bar{K}$. This inequality follows from the fact that

$$\frac{dF((1 - \alpha) / \bar{K}) \bar{K}}{d\bar{K}} = F\left(\frac{1 - \alpha}{\bar{K}}\right) - F'\left(\frac{1 - \alpha}{\bar{K}}\right) \frac{1 - \alpha}{\bar{K}} < 0,$$

and the sign of the latter inequality holds since $\bar{a}_2 > \hat{a}$ and at the end of the proof of Proposition 2 we showed that $F'(a)a > F(a)$ iff $a > \hat{a}$. Hence problem (30) - and so $[\mathbf{IP}(\bar{K})]$ - attains a minimum at \bar{a}_2 .

It is immediate to see that, when $\bar{K} < K^c$, the solution of problem (30), replicated for each $i = 1, \dots, \bar{K} + 1$, also constitutes a solution of problem $[\mathbf{P}(\bar{K})]$ since the additional constraints present in the latter problem are satisfied at this solution. Hence we have $V(\bar{K}) = D(\bar{K})$. In contrast, for $\bar{K} \geq K^c$ the solution of (30), replicated for $i = 1, \dots, \bar{K} + 1$, violates the last constraint of problem $[\mathbf{P}(\bar{K})]$ and hence in that case we have

$V(\bar{K}) \geq D(\bar{K})$. Also, by continuity of $V(\bar{K})$ with respect to ε , $\lim_{\varepsilon \rightarrow 0} V(\bar{K}) = D(\bar{K}) = K^c F((1-\alpha)/K^c)$.

We turn then our attention to problem **[SP]**. Having assumed that $N < 2(\hat{K} + 1)$ we claim that at a social optimum we have two components of the same size $N/2 < \hat{K} + 1$. We cannot in fact have a component of size $K_i > \hat{K}$ and another one of size $K_j < \hat{K}$ at a solution of **[SP]**: reducing K_i and increasing K_j by the same amount is going to reduce average default across the two components, since we showed above that $V(\bar{K}) = D(\bar{K}) = F((1-\alpha)/\bar{K}) \bar{K}$ is strictly decreasing for $\bar{K} < \hat{K}$ and for ε sufficiently small $V(\bar{K})$ is essentially constant, as argued above, for $\bar{K} > \hat{K}$. Also, we cannot have a component of size $K_i < \hat{K}$ and another one of size $K_j < \hat{K}$ with $K_i \neq K_j$, by the convexity of $V(\cdot)$ for $\bar{K} < \hat{K}$. We have in fact

$$\begin{aligned} V''(\bar{K}) &= -F'((1-\alpha)/\bar{K})(1-\alpha)/\bar{K}^2 + F'((1-\alpha)/\bar{K})(1-\alpha)/\bar{K}^2 + F''((1-\alpha)/\bar{K})(1-\alpha)^2/\bar{K}^2 \\ &= F''((1-\alpha)/\bar{K})(1-\alpha)^2/\bar{K}^2 > 0, \end{aligned}$$

by the convexity of $F(\cdot)$ at $\bar{a}_2 > \hat{a}$. Given the assumption that $N < 2(\hat{K} + 1)$ we cannot have two components both of size bigger than \hat{K} . We are then only left with the possibility of having components all of the same size $K^* < \hat{K}$. Since a solution exists this establishes the result. \square

Lemma For $2 \leq \gamma_1 \leq 3$ there is a unique value $a'_2 \in (\bar{a}_2, (1-\alpha)/(\bar{K}-1))$ satisfying the equation $\Pi(a_2) \equiv F'(a_2) - F'(1-\alpha - (\bar{K}-1)a_2) = 0$.

Proof. We have shown in the proof of Proposition 2 that $F'(\cdot)$ is increasing for $a_2 > a^{**}$ and decreasing for $a_2 < a^{**}$, also that $\hat{a} > a^{**}$. Since $\bar{a}_2 > \hat{a}$, $F'(\cdot)$ is increasing for all $a'_2 \in (\bar{a}_2, (1-\alpha)/(\bar{K}-1)]$, while $F'(1-\alpha - (\bar{K}-1)a_2)$ is increasing in a_2 for a_2 close to $(1-\alpha)/(\bar{K}-1)$ and decreasing³⁷ for a_2 close to \bar{a}_2 . Since $\Pi(\bar{a}_2) = 0$, this implies that $\Pi(a_2) > 0$ for a_2 close to \bar{a}_2 . On the other hand, since $F'(1-\alpha - (\bar{K}-1)a_2)$ tends to infinity as a_2 tends to $(1-\alpha)/(\bar{K}-1)$ (hence $a_1 \rightarrow 0$) while $F'(a_2)$ remains bounded, $\Pi(a_2)$ must be negative when a_2 is close to $(1-\alpha)/(\bar{K}-1)$. Thus, there must be at least one value $a'_2 \in (\bar{a}_2, (1-\alpha)/(\bar{K}-1))$ such that $\Pi(a'_2) = 0$.

We now show that, for $2 \leq \gamma_1$, the function $\Pi(a_2)$ changes concavity in the range $(\bar{a}_2, (1-\alpha)/(\bar{K}-1))$ at most once. We have in fact

$$\begin{aligned} \Pi''(a_2) &= p\gamma_1(\gamma_1-1)(\gamma_1-2)a_2^{\gamma_1-3} \left(\frac{L}{R-M} \right)^{\gamma_1} + (1-p)\gamma_2(\gamma_2-1)(\gamma_2-2)a_2^{\gamma_2-3} \left(\frac{L}{R-M} \right)^{\gamma_2} - \\ &\quad (\bar{K}-1)^2 \left(\begin{aligned} &p\gamma_1(\gamma_1-1)(\gamma_1-2)(1-\alpha - (\bar{K}-1)a_2)^{\gamma_1-3} \left(\frac{L}{R-M} \right)^{\gamma_1} \\ &+ (1-p)\gamma_2(\gamma_2-1)(\gamma_2-2)(1-\alpha - (\bar{K}-1)a_2)^{\gamma_2-3} \left(\frac{L}{R-M} \right)^{\gamma_2} \end{aligned} \right). \end{aligned}$$

³⁷In this case a_1 is in fact close to a_2 and hence greater than \hat{a} , while in the previous case a_1 is close to 0.

It is convenient to collect terms and rewrite this expression as

$$\Pi''(a_2) = G_1(a_2) - (\bar{K} - 1)^2 G_2(a_2)$$

with

$$G_1(a_2) \equiv p\gamma_1(\gamma_1 - 1)(\gamma_1 - 2)a_2^{\gamma_1-3} \left(\frac{\underline{L}}{R-M} \right)^{\gamma_1} + (1-p)\gamma_2(\gamma_2 - 1)(\gamma_2 - 2)a_2^{\gamma_2-3} \left(\frac{\underline{L}}{R-M} \right)^{\gamma_2}$$

and

$$\begin{aligned} G_2(a_2) \equiv & p\gamma_1(\gamma_1 - 1)(\gamma_1 - 2)(1 - \alpha - (\bar{K} - 1)a_2)^{\gamma_1-3} \left(\frac{\underline{L}}{R-M} \right)^{\gamma_1} + \\ & (1-p)\gamma_2(\gamma_2 - 1)(\gamma_2 - 2)(1 - \alpha - (\bar{K} - 1)a_2)^{\gamma_2-3} \left(\frac{\underline{L}}{R-M} \right)^{\gamma_2}. \end{aligned}$$

Since $\gamma_1 \geq 2$, $(\gamma_1 - 1)(\gamma_1 - 2) \geq 0$, so that both functions $G_1(a_2)$, $G_2(a_2)$ are always positive. Also, $\gamma_1 \leq 3$ and $1 - \alpha - (\bar{K} - 1)a_2 \geq 0$ imply that $G_1(a_2)$ is decreasing and $G_2(a_2)$ is increasing with $\lim_{a_2 \rightarrow (1-\alpha)/(\bar{K}-1)} G_2(a_2) = \infty$. It then follows that $\Pi''(a_2)$ is decreasing for all $a_2 \in (\bar{a}_2, (1 - \alpha) / (\bar{K} - 1))$ and $\lim_{a_2 \rightarrow (1-\alpha)/(\bar{K}-1)} \Pi''(a_2) = -\infty$. This implies that there can be at most one value $\tilde{a}_2 \in (\bar{a}_2, (1 - \alpha) / (\bar{K} - 1))$ at which we have $\Pi''(\tilde{a}_2) = 0$, that is the function $\Pi(a_2)$ can change from convex to concave at most once in the range considered.

We have seen that $\Pi(a_2)$ is increasing in the leftmost part of the interval $(\bar{a}_2, (1 - \alpha) / (\bar{K} - 1))$ and that this part of the interval is also the only region where $\Pi(a_2)$ can be convex. Then it must be the case that either $\Pi(a_2)$ is concave in the whole range, or it is convex in the leftmost part of the interval $(\bar{a}_2, (1 - \alpha) / (\bar{K} - 1))$ and then concave.

Let us first look at the case where $\Pi(a_2)$ is concave in the whole range. Since $\Pi(a_2)$ is increasing at \bar{a}_2 and decreasing as $a_2 \rightarrow (1 - \alpha) / (\bar{K} - 1)$, it has a single local and global maximum $\bar{\bar{a}}_2$, to the right of which $\Pi(a_2)$ is decreasing, so it can cross the horizontal axis only once.

Consider next the case where $\Pi(a_2)$ is convex in the leftmost part of the interval $(\bar{a}_2, (1 - \alpha) / (\bar{K} - 1))$ and then concave. Since $\Pi(a_2)$ is continuously differentiable it cannot change from being increasing to decreasing without changing first from being convex to concave. This means that at the unique inflexion

point \tilde{a}_2 $\Pi(\cdot)$ is still increasing and is so increasing in the interval (\bar{a}_2, \tilde{a}_2) , so $\Pi(\cdot)$ cannot equal zero in that interval. In the remaining interval $(\tilde{a}_2, (1 - \alpha) / (\bar{K} - 1))$ $\Pi(\cdot)$ is concave, and then we can apply the argument of the previous paragraph. In both cases we have so shown that there can only be one value $a'_2 \in (\bar{a}_2, (1 - \alpha) / (\bar{K} - 1))$ such that $\Pi(a'_2) = 0$. \square

Proof of Proposition 4

Under the conditions stated in the proposition, the objective function of problem [IP] takes the following form:

$$\sum_{j \neq i} (pF(a_{ij}) + (1 - p)I(a_{ij} > (R - M) / L_1)), \quad (33)$$

where $I(\cdot)$ is the indicator function. First, note that at a solution of [IP] we cannot have two values $a_{ij} \neq a_{ik}$ with $a_{ij} \leq (R - M) / L_1$, $a_{ik} \leq (R - M) / L_1$. This is immediate because otherwise we can always find an improvement: the choice $a'_{ij} = a'_{ik} = (a_{ij} + a_{ik}) / 2$ is in fact feasible, does not affect the value of the second part of the objective function in (33), and by the convexity of $F(\cdot)$ decreases the value of the first part, where $F(\cdot)$ appears. By a symmetric argument it also follows that there cannot be two values $a_{ij} \neq a_{ik}$ with $a_{ij} > (R - M) / L_1$, $a_{ik} > (R - M) / L_1$.

Condition (16) implies that we must have³⁸ $a_{ik} > (R - M) / L_1$ for at least some $k \neq i$. It then follows that there can be no value $a_{ij} < (R - M) / L_1$. If $a_{ij} < (R - M) / L_1$, the alternative choice $a'_{ij} = (R - M) / L_1$ and $a'_{ik} = a_{ik} - ((R - M) / L_1 - a_{ij})$ would allow an improvement: it is feasible, does not affect the second part of the objective function and by convexity of $F(\cdot)$ decreases the first part. Hence there can be no term $a_{ij} = 0$, proving there has to be a single component.

From the above arguments it follows that at a solution of [IP] we have $N - H - 1 \geq 0$ terms with exposure $a_{ij} = (R - M) / L_1$ and the remaining $H \geq 0$ terms with equal exposure $a > (R - M) / L_1$, such that

$$(N - H - 1) \frac{R - M}{L_1} + Ha = 1 - \alpha. \quad (34)$$

Solving this equation for a in terms of H and substituting into the objective function, we obtain that the

³⁸If $a_{ij} \leq (R - M) / L_1$ for all j feasibility is violated.

optimal value of H solves the following problem³⁹:

$$\min_{N-1 \geq H} H(1-p) + pHF \left(\frac{1}{H} \left(1 - \alpha - (R-M) \frac{N-H-1}{L_1} \right) \right) + p(n-H-1)F \left(\frac{R-M}{L_1} \right)$$

The first order condition for an interior solution is

$$(1-p) + pF(a) + pF'(a) \left(-a + \frac{R-M}{L_1} \right) - pF \left(\frac{R-M}{L_1} \right) = 0,$$

while the condition for a corner solution at $H = N - 1$ is:

$$(1-p) + p \left(F \left(\frac{1-\alpha}{N-1} \right) - F \left(\frac{R-M}{L_1} \right) \right) + pF' \left(\frac{1-\alpha}{N-1} \right) \left(-\frac{1-\alpha}{N-1} + \frac{R-M}{L_1} \right) \leq 0.$$

Under condition (17) the latter inequality never holds, hence the solution is interior, with exposure $(R-M)/L_1$ to a positive number of, but not all, other firms, and higher exposure a to the remaining firms.

Since the solution we obtained is symmetric and features a single component, it satisfies the additional constraints present in problem [SP], (9) and (10), and hence it also solves problem [SP]. \square

Proof of Proposition 6

We show first that for each firm j the size of the component $\mathcal{C}_j(\mathbf{S})$ to which j belongs has size $|\mathcal{C}_j(\mathbf{S})| + 1 \leq \hat{K} + 1$, where \hat{K} is the value obtained at the solution of the individual problem [IP], characterized in Proposition 2. That is, no component in the *CPE* has size larger or equal to the individually optimal one. Suppose not: $|\mathcal{C}_j(\mathbf{S})| > \hat{K}$ for some j . A subset $W \subset \mathcal{C}_j(\mathbf{S})$ formed of $\hat{K} + 1$ firms can then profitably deviate by severing the linkages with the other firms in $\mathcal{C}_j(\mathbf{S})$ and attain a lower probability of default, since $V(K)$ attains indeed a minimum at \hat{K} .

Next, we show that at a *CPE* we have $|\mathcal{C}_j(\mathbf{S})| < \hat{K}$ for some j and $|\mathcal{C}_i(\mathbf{S})| = \hat{K}$ for all other $\mathcal{C}_i(\mathbf{S}) \neq \mathcal{C}_j(\mathbf{S})$. We proceed again by contradiction. Suppose there were two distinct components $\mathcal{C}_i(\mathbf{S}), \mathcal{C}_j(\mathbf{S})$ such that $0 < |\mathcal{C}_j(\mathbf{S})| \leq |\mathcal{C}_i(\mathbf{S})| < \hat{K}$. Pick then a subset of firms belonging to the first component $W' \subset \mathcal{C}_i(\mathbf{S})$ so that the set given by the union of W' and $\mathcal{C}_j(\mathbf{S})$ has $\hat{K} + 1$ firms in it. The firms in $W' \cup \mathcal{C}_j(\mathbf{S})$ can deviate by severing the linkages of the firms in W' with the other firms in $\mathcal{C}_i(\mathbf{S}) \setminus W'$ and forming new linkages with all the firms in $\mathcal{C}_j(\mathbf{S})$. By so doing each firm in $W' \cup \mathcal{C}_j(\mathbf{S})$ attains a lower level of its probability of default, equal to $V(\hat{K})$. Hence all firms involved benefit from the deviation, a contradiction. This contradiction

³⁹Note that at a solution of (34) we must have $H > 0$, hence the non-negativity constraint on this variable never binds and can be ignored.

establishes the above claim and completes so the proof of the Proposition. \square

Proof of Proposition 7

Suppose the system is arranged into C components of size $K^c + 1$, for $c = 1, \dots, C$, and in each component c there are $|N_t^c|$ firms of type t . Using (18), the expected number of firms in the system that default when indirectly hit by a b shocks (and when the linkages within components are optimally chosen) is then

$$\sum_{c=1}^C (K^c + 1) V \left(K^c, (|N_t^c|, \Phi_t)_{t=1}^T \right) \geq \sum_{c=1}^C (K^c + 1) \sum_{t=1}^T \frac{|N_t^c|}{K^c + 1} V(K^c, \Phi_t).$$

Exchanging the summation indices the term on the right hand side of the above inequality satisfies the following

$$\begin{aligned} \sum_{c=1}^C \sum_{t=1}^T |N_t^c| V(K^c, \Phi_t) &\geq \sum_{t=1}^T \sum_{c=1}^C |N_t^c| V(\hat{K}_t, \Phi_t) \\ &= \sum_{t=1}^T |N_t| V(\hat{K}_t, \Phi_t) \end{aligned}$$

Note that, since $|N_t|$ is assumed to be a multiple of $\hat{K}_t + 1$, $\sum_{t=1}^T |N_t| V(\hat{K}_t, \Phi_t)$ is also the expected number of defaults in the system if every firm of type t is part of a component with \hat{K}_t other firms all of the same type t (with the pattern of linkages set at the values obtained from a solution of problem [IP] when the distribution of the b shocks is Φ_t for all firms). The above inequality shows that a segmentation of the system into homogenous components allows to attain a lower number of expected defaults than any other pattern of segmentation. Moreover, as long as the value of \hat{K}_t and/or the optimal intensity of linkages in a component vary non-trivially with the firms' type t , the above inequalities are strict, hence all optimal financial structures exhibit segmentation into homogenous components. \square

Proof of Proposition 8

Note that the expression in (19) of the expected fraction of firms defaulting in a component with $|N_1^c|$ unit sized firms and $|N_\beta^c|$ large firms can be conveniently rewritten as follows:

$$\frac{1}{K^c + 1} \left[\sum_{i \in N_1^c} \left(\sum_{j \in N_1^c/i} F(a_{ij}) + \beta \sum_{j \in N_\beta^c} F(a_{ij}) \right) + \sum_{i \in N_\beta^c} \beta \left(\sum_{j \in N_1^c} F\left(\frac{a_{ij}}{\beta}\right) + \sum_{j \in N_\beta^c/i} F\left(\frac{a_{ij}}{\beta}\right) \right) \right].$$

In addition, the balance condition (11) is, in this case:

$$\begin{aligned} \sum_{j \in N_1^c \setminus i} a_{ij} + \sum_{j \in N_\beta^c} a_{ij} \beta &= 1 - \alpha, \quad \forall i \in N_1^c \\ \sum_{j \in N_1^c \setminus i} a_{ij} \frac{1}{\beta} + \sum_{j \in N_\beta^c \setminus i} a_{ij} &= 1 - \alpha, \quad \forall i \in N_\beta^c. \end{aligned}$$

Hence, if we proceed to the following change of variable $\tilde{a}_{ij} \equiv a_{ij}/\beta$ for $i \in N_\beta^c$ and all j we see that the minimization problem whose solution yields $V(K^c, |N_1^c|, |N_\beta^c|)$ becomes the same as problem $[\mathbf{P}(K^c)]$, subject to the additional constraint that at least β terms in each row and column of A_{K^c} take the same value. Note that the latter constraint is always satisfied at a solution of problem $[\mathbf{P}(K^c)]$ in all the cases considered in Section 3 (Propositions 1, 2, 4), whose solution yields $V(K^c)$. \square

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